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Fractional standard map

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ABSTRACT

Properties of the phase space of the standard map with memory are investigated. This map was obtained from a kicked fractional differential equation. Depending on the value of the map parameter and the fractional order of the derivative in the original differential equation, this nonlinear dynamical system demonstrates attractors (fixed points, stable periodic trajectories, slow converging and slow diverging trajectories, ballistic trajectories, and fractal-like structures) and/or chaotic trajectories. At least one type of fractal-like sticky attractors in the chaotic sea was observed.

1. Introduction

The standard map (SM) can be derived from the differential equation describing kicked rotator. The description of many physical systems and effects (Fermi acceleration, comet dynamics, etc.) can be reduced to the studying of the SM [1]. The SM provides the simplest model of the universal generic area preserving map and it is one of the most widely studied maps. The topics examined include fixed points, elementary structures of islands and a chaotic sea, and fractional kinetics [1–3].

It was recently realized that many physical systems, including systems of oscillators with long range interaction [4,5], non-Markovian systems with memory ([6, Chapter 10], [7–11]), fractal media [12], etc., can be described by the fractional differential equations (FDE) [6,13,14]. As with the usual differential equations, the reduction of FDEs to the corresponding maps can provide a valuable tool for the analysis of the properties of the original systems. As in the case of the SM, the fractional standard map (FSM), derived in [15] from the fractional differential equation describing a kicked system, is perhaps the best candidate to start a general investigation of the properties of maps which can be obtained from FDEs.

As it was shown in [15], maps that can be derived from FDEs are of the type of discrete maps with memory. One-dimensional maps with memory, in which the present state of evolution depends on all past states, were studied previously in [16–21]. They were not derived from differential equations. Most results were obtained for the generalizations of the logistic map.

In the physical systems the transition from integer order time derivatives to fractional (of a lesser order) introduces additional damping and is similar in appearance to additional friction [6,22]. Accordingly, in the case of the FSM we may expect transformation of the islands of stability and the accelerator mode islands into attractors (points, attracting trajectories, strange attractors). Because the damping in systems with fractional derivatives is based on the internal causes different from the external forces of friction [22, 23], the corresponding attractors are also different from the attractors of the regular systems with friction and are called fractional attractors [22]. Even in one-dimensional cases [16–21] most of the results were obtained numerically. An additional dimension makes the problem even more complex and most of the results in the present Letter were obtained numerically.

2. FSM, initial conditions

The standard map in the form

$$p_{n+1} = p_n - K \sin x_n,$$
$$x_{n+1} = x_n + p_{n+1} \pmod{2\pi} \quad (1)$$

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can be derived from the differential equation

$$\ddot{x} + K\sin(x)\sum_{n=0}^{\infty}\delta\left(\frac{t}{T} - n\right) = 0. \tag{2}$$

By replacing the second-order time derivative in Eq. (2) with the Riemann–Liouville derivative ${}_0D_t^\alpha$ one obtains a fractional equation of motion in the form

$${}_0D_t^\alpha x + K\sin(x)\sum_{n=0}^{\infty}\delta\left(\frac{t}{T} - n\right) = 0 \quad (1 < \alpha \leqslant 2), \tag{3}$$

where

$${}_0D_t^\alpha x(t) = D_t^m {}_0I_t^{m-\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t \frac{x(\tau)\,d\tau}{(t-\tau)^{\alpha-m+1}} \quad (m-1 < \alpha \leqslant m), \tag{4}$$

$D_t^m = d^m/dt^m$, and ${}_0I_t^\alpha$ is a fractional integral. The initial conditions for (3) are

$$\left({}_0D_t^{\alpha-1}x\right)(0+) = p_1,$$
$$\left({}_0D_t^{\alpha-2}x\right)(0+) = b. \tag{5}$$

The Cauchy type problem (3) and (5) is equivalent to the Volterra integral equation of the second kind [24–26]

$$x(t) = \frac{p_1}{\Gamma(\alpha)}t^{\alpha-1} + \frac{b}{\Gamma(\alpha-1)}t^{\alpha-2} - \frac{K}{\Gamma(\alpha)}\int_0^t \frac{\sin[x(\tau)]\sum_{n=0}^{\infty}\delta(\frac{\tau}{T} - n)\,d\tau}{(t-\tau)^{1-\alpha}}. \tag{6}$$

Defining the momentum as

$$p(t) = {}_0D_t^{\alpha-1}x(t), \tag{7}$$

and performing integration in (6) one can derive the equation for the FSM in the form (for the thorough derivation see [26])

$$p_{n+1} = p_n - K\sin x_n, \tag{8}$$

$$x_{n+1} = \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n} p_{i+1}V_\alpha(n-i+1) + \frac{b}{\Gamma(\alpha-1)}(n+1)^{\alpha-2} \quad (\mathrm{mod}\, 2\pi), \tag{9}$$

where

$$V_\alpha(m) = m^{\alpha-1} - (m-1)^{\alpha-1}. \tag{10}$$

Here it is assumed that $T = 1$ and $1 < \alpha \leqslant 2$. The form of Eq. (9) which provides a more clear correspondence with the SM ($\alpha = 2$) in the case $b = 0$ is presented in Section 4 (Eq. (31)).

The second initial condition in (5) can be written as

$$\left({}_0D_t^{\alpha-2}x\right)(0+) = \lim_{t\to 0+} {}_0I_t^{2-\alpha}x(t) = \lim_{t\to 0+}\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{x(\tau)\,d\tau}{(t-\tau)^{\alpha-1}} = b \quad (1 < \alpha \leqslant 2), \tag{11}$$

which requires $b = 0$ in order to have a solution bounded at $t = 0$ for $\alpha < 2$. The assumption $b = 0$ leads to the FSM equations which in the limiting case $\alpha = 2$ coincide with the equations for the standard map under the condition $x_0 = 0$.

In this Letter the FSM is taken in the form derived in [15] which coincides with (8) and (9) if $b = 0$. It is also assumed that $x_0 = 0$ and the results can be compared to those obtained for the SM with $x_0 = 0$ and arbitrary p_0. As a test, for the SM and for the FSM with $\alpha = 2$ and the same initial conditions numerical calculations show that phase portraits look identical.

System of Eqs. (8) and (9) can be considered either in a cylindrical phase space (x mod 2π) or in unbounded phase space. The second case is convenient to study transport. The trajectories in the second case are easily related to the first case. The FSM has no periodicity in p (the SM does) and cannot be considered on a torus.

3. Stable fixed point

The SM has stable fixed points at $(0, 2\pi n)$ for $K < K_c = 4$. It is easy to see that point $(0, 0)$ is also a fixed point for the FSM. Direct computations using (8) and (9) demonstrate that for the small initial values of p_0 there is a clear transition from the convergence to the fixed point to divergence when the value of the parameter K crosses the curve $K = K_c(\alpha)$ on Fig. 1(a) from smaller to larger values.

The following system describes the evolution of trajectories near fixed point $(0, 0)$

$$\delta p_{n+1} = \delta p_n - K\delta x_n, \tag{12}$$

$$\delta x_{n+1} = \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n}\delta p_{i+1}V_\alpha(n-i+1). \tag{13}$$

The solution can be found in the form

$$\delta p_n = p_0\sum_{i=0}^{n-1} p_{n,i}\left(\frac{2}{V_{\alpha l}}\right)^i\left(\frac{V_{\alpha l}K}{2\Gamma(\alpha)}\right)^i \quad (n > 0), \tag{14}$$

$$\delta x_n = \frac{p_0}{\Gamma(\alpha)}\sum_{i=0}^{n-1} x_{n,i}\left(\frac{2}{V_{\alpha l}}\right)^i\left(\frac{V_{\alpha l}K}{2\Gamma(\alpha)}\right)^i \quad (n > 0). \tag{15}$$

The origin of the terms in parentheses, as well as the definition

$$V_{\alpha l} = \sum_{k=1}^{\infty}(-1)^{k+1}V_\alpha(k) \tag{16}$$

will become clear in Section 5. Eqs. (12)–(16) lead to the following iterative relationships

$$x_{n+1,i} = -\sum_{m=i}^{n}(n-m+1)^{\alpha-1}x_{m,i-1} \quad (0 < i \leqslant n), \tag{17}$$

$$p_{n+1,i} = -\sum_{m=i}^{n}x_{m,i-1} \quad (0 < i < n) \tag{18}$$

with the initial and boundary conditions

$$p_{n+1,n} = x_{n+1,n} = (-1)^n, \qquad p_{n+1,0} = 1,$$
$$x_{n+1,0} = (n+1)^{\alpha-1}. \tag{19}$$

From (17) and (18) it is clear that the series (14) and (15) are alternating and it is natural to apply the Dirichlet's test to verify their convergence. This can be done by considering the totals

$$S_n = \sum_{i=0}^{n-1} x_{n,i}\left(\frac{2}{V_{\alpha l}}\right)^i, \tag{20}$$