Physics Letters A 373 (2009) 4322–4327

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

# A new family of evolution water-wave equations possessing two-soliton solutions

Yulia Yu. Bagderina

*Institute of Mathematics with Computer Centre of Russian Academy of Sciences, 112 Chernyshevsky str., Ufa 450008, Russia*

ARTICLE INFO

*Article history:*
Received 29 February 2008
Received in revised form 13 August 2009
Accepted 15 September 2009
Available online 22 September 2009
Communicated by A.P. Fordy

*PACS:*
02.30.Jr
04.20.Jb

*Keywords:*
Evolution equation
Invariant manifold
Two-soliton solution

ABSTRACT

We find a new family of fifth-order water-wave equations having common invariant manifold of the fourth order. These evolution equations are nonintegrable except for two cases corresponding to the Sawada–Kotera and Kaup–Kupershmidt equations. The invariant manifold of the family is an autonomous equation F-VI from the Cosgrove's classification of fourth-order ODEs having the Painlevé property. Two-parameter solutions of the equation F-VI allow to find two-soliton solutions for this family of evolution equations.



## 1. Introduction

As is well known, the Kaup–Kupershmidt (KK) equation [1]

$$v_t = v_{xxxxx} - 15 v v_{xxx} - \frac{75}{2} v_x v_{xx} + 45 v^2 v_x \tag{1}$$

and the Sawada–Kotera (SK) equation [2,3]

$$v_t = v_{xxxxx} - 30 v v_{xxx} - 30 v_x v_{xx} + 180 v^2 v_x \tag{2}$$

have an invariant manifold [3,4]

$$v_{xxxx} - 18 v v_{xx} - 9 v_x^2 + 24 v^3 + K_2\big(v_{xx} - 3k v^2\big) + K_1 = 0,$$

where $k = 2$ for the KK equation and $k = 1$ for the SK equation, $K_1$, $K_2$ are arbitrary constants. Translation $v = u + K_2/18$ transforms it into an autonomous case (with $\alpha = (-1)^k k K_2$, $\beta = (6K_2^3 + \alpha^3)/972 - K_1$)

$$u_{xxxx} = 18 u u_{xx} + 9 u_x^2 - 24 u^3 + \alpha u^2 + \frac{\alpha^2}{9} u + \beta \tag{3}$$

of the equation F-VI from the classification of Cosgrove [5] of the fourth-order ordinary differential equations (ODEs) having the Painlevé property. Other ODEs in this classification, the equations F-III, F-IV and autonomous equation F-V, represent the group-invariant reduction of the KK, SK and fifth-order Korteweg–de Vries (KdV) equations respectively. In [5] the general solution of equations F-III, ..., F-VI has been constructed in terms of hyperelliptic functions of genus two. Particular two-parameter solutions of these ODEs allow to find two-soliton solutions for evolution equations of the corresponding integrable hierarchies.

According to [6], an ODE

$$\Phi\big(x, u, u_x, \dots, \partial^k u/\partial x^k\big) = 0 \tag{4}$$

defines the invariant manifold of the evolution equation

$$u_t = F\big(x, u, u_x, \dots, \partial^p u/\partial x^p\big), \tag{5}$$

if the relation

$$X\Phi|_{[\Phi]=0} = 0, \qquad X = F\partial_u + \sum_{j=1}^{k} D_x^j F \partial_{u_j} \tag{6}$$

holds. Here $u_j = \partial^j u/\partial x^j$, $D_x$ is the operator of total differentiation with respect to $x$, $[\Phi] = 0$ is the manifold defined by Eq. (4) and its consequences $D_x^j \Phi = 0$, $j = 1, \dots, p$. If the solution

$$u(x) = U(x, c_1, \dots, c_k), \quad c_1, \dots, c_k = \text{const}$$

of ODE (4) is found, then evolution equation (5) have a solution of the form

$$u(t, x) = U\big(x, c_1(t), \dots, c_k(t)\big).$$

The substitution of this function into Eq. (5) yields the system of first-order ODEs in $c_1(t), \dots, c_k(t)$. Thus, finding the exact solutions to evolution equations with the use of invariant manifolds (4) can be regarded as a generalized separation of variables.

*E-mail address:* yulya@mail.rb.ru.

0375-9601/$ – see front matter 
doi:10.1016/j.physleta.2009.09.051

In [7,8] the linear invariant manifolds were applied to constructing the solutions of the form of finite sums

$$u(t,x) = c_1(t) f_1(x) + c_2(t) f_2(x) + \cdots + c_k(t) f_k(x).$$

But most of the solutions of physical interest ($N$-soliton, rational, etc.) have the form

$$u(t,x) = \frac{c_0(t) f_0(x) + c_1(t) f_1(x) + \cdots + c_k(t) f_k(x)}{c_0(t) g_0(x) + c_1(t) g_1(x) + \cdots + c_k(t) g_k(x)}.$$

To find them, in a recent paper [9] we have proposed to use the nonlinear invariant manifolds (4) depending on $u, u_x, \ldots, \partial^k u/\partial x^k$ like the generalized Riccati equations [10]. This approach was illustrated with the equation

$$v_t + v_{xxxxx} - a v v_{xxx} + 2(a-30) v_x v_{xx} + v_{xxx} - c v v_x = 0, \quad a, c = \text{const}, \tag{7}$$

which describes the long waves in shallow water [11]. It has been proved in [12] that Eq. (7) is nonintegrable, it does not pass the WTC-test [13]. We have found [9] that for $a = 10$ Eq. (7) possesses an invariant manifold

$$v_{xxxx} - 18 v v_{xx} - 9 v_x^2 + 24 v^3 + \frac{9-c}{5} v_{xx} + \frac{9}{10}(c-8) v^2 + (c-6)(c-12)\frac{v}{100} + K_1 = 0.$$

Translation $v = u + (9-c)/90$ turns this ODE into Eq. (3) with $\alpha = -c/10$, $\beta = 3(c-9)(8c^2 - 171c + 648)/90^3 - K_1$. Hence there exist three evolution equations (1), (2) and (7) having the invariant manifolds reduced to the same equation F-VI.

Then an inverse problem arises: to describe all evolution equations of the form

$$u_t = u_{xxxxx} + A_1 u u_{xxx} + A_2 u_x u_{xx} + A_3 u^2 u_x + A_4 u_{xxx} + A_5 u u_x, \quad A_1, \ldots, A_5 = \text{const}, \tag{8}$$

having the invariant manifolds, which coincide with ODEs F-III, ..., F-VI from [5]. Eq. (8) admits five-parameter equivalence group of time and space translations, two dilations and a Galilean transformation

$$\tau = a_3^{-5} t + a_5, \qquad z = a_3^{-1}\left(x + \left(a_1^2 A_3 - a_1 A_5\right)t\right) + a_4, \quad v = a_3^2 a_2^{-1}(u + a_1), \quad a_2, a_3 \neq 0, \tag{9}$$

which turn Eq. (8) into an equation of the same form

$$v_\tau = v_{zzzzz} + a_2 A_1 v v_{zzz} + a_2 A_2 v_z v_{zz} + a_2^2 A_3 v^2 v_z + a_3^2(A_4 - a_1 A_1) v_{zzz} + a_3^2 a_2 (A_5 - 2a_1 A_3) v v_z.$$

Relation (6) for ODE (3) and evolution equation (8) becomes

$$2(54A_4 + 5A_5 + 10\alpha) u_{xx} u_{xxx} + \left(342 A_4 + 27 A_5 + \alpha(8A_1 + 2A_2 + 270)\right) u_x^3 - 120(A_1 + 2A_2 + 90) u^3 u_{xxx} - 240(9A_1 + A_3 + 90) u^4 u_x + \cdots = 0$$

(here we omit the remaining terms of the polynomial) and vanishes with $A_1 = -A_3/9 - 10$, $A_2 = A_3/18 - 40$, $A_4 = 5\alpha \times (A_3/18 - 4)/18$, $A_5 = \alpha(10 - A_3/6)$. Hence equation F-VI provides the invariant manifold for a whole family of evolution equations

$$u_t = u_{xxxxx} - 10(b+1) u u_{xxx} + 5(b-8) u_x u_{xx} + 90 b u^2 u_x + \frac{5}{18}\alpha(5b-4) u_{xxx} + 5\alpha(2-3b) u u_x, \quad b = \text{const}. \tag{10}$$

Similar computation shows that equations F-III, F-IV and F-V define the invariant manifold only for the KK, SK and KdV equations respectively. Solving these ODEs one can find traveling wave solutions of the corresponding fifth-order evolution equations.

Eq. (10) coincides (up to transformation (9), where $a_1 = \alpha(1-b)/18$, $a_2 = 1$, $a_3 = 1$, $a_4 = 0$, $a_5 = 0$) with the KK equation (1) in the case of $b = 1/2$ and with the SK equation (2), when $b = 2$. It is nonintegrable for other values of $b$. In [14] WTC-type expansions with logarithmic terms have been constructed for a class of equations (8). It was established there that the solutions of 27 equations of the form (8) can be expanded into the series with or without logarithms with four nonnegative resonances. Seven of these equations are involved into the family (10), when $b$ equals to 2, 1/2, 2/5, 4/5, 4, 13/10 or 29/10. The class of evolution equations (8) occurs in water-wave models and several other applications (see [15] for many references).

It was shown in [16] for Eq. (8) with the parameters $A_1 = 8A - 2B$, $A_2 = 4A - 6B$, $A_3 = -20AB$, $A_4 = 0$, $A_5 = 0$ that three cases when it is a soliton equation (SK, KK or KdV) are closely related to the only cases when the cubic Hénon–Heiles Hamiltonian system passes the Painlevé test. This equation

$$v_\tau = \left(v_{zzzz} + (8A - 2B) v v_{zz} - 2(A+B) v_z^2 - \frac{20}{3} A B v^3\right)_z \tag{11}$$

is related to Eq. (10) with $\alpha = 0$ by transformation (9), iff the equalities

$$10 a_2 (b+1) = 2B - 8A, \qquad 5a_2(b-8) = 4A - 6B, \qquad 90 a_2^2 b = -20AB$$

hold. With $4A = a_2(2-7b)$, $B = a_2(7-2b)$ obtained from first two equalities the third one becomes $a_2^2(b-2)(2b-1) = 0$. Therefore, integrable SK and KK equations are the only equations (8) involved into both families (10) and (11).

The outline of this Letter is as follows. In Section 2 other nonlinear invariant manifolds of Eq. (10) are found from criterion (6). Using the solutions of these ODEs, the traveling wave solutions to Eq. (10) are constructed. In Section 3 we study two-soliton and oscillating solutions of Eq. (10) arising from two-parameter solutions of the equation F-VI. In Section 4 we construct bilinear form of Eq. (10) following the approach applied in [17] to the KK equation. Next we show that two-soliton solution obtained in Section 3 may be found also using Hirota's bilinear method [18].

## 2. Traveling wave solutions of Eq. (10)

Invariant manifolds of Eq. (10) are found from the criterion (6), where we set the right-hand side of Eq. (10) in place of $F$. It turns out that, in addition to ODE (3), Eq. (10) has the following nonlinear invariant manifolds:

$$u_{xx} - 6u^2 - \frac{\alpha}{3} u + K_1 = 0, \tag{12}$$

$$u_{xx} - \frac{3}{2} b u^2 + K_2 u + K_1 = 0, \quad K_2 = \alpha \frac{5b^2 - 22b + 12}{12(b-5)}, \quad b \neq 5; 0, \tag{13}$$

$$(u - K_2)\left(u_{xx} - 3u^2 + \frac{\alpha}{3} u + K_1\right) - \frac{3}{4} u_x^2 = 0, \quad K_2 = \frac{5}{36}\alpha, \tag{14}$$

$$u_{xxxx} - 10(b+1) u u_{xx} + \frac{15}{2}(b-2) u_x^2 + 30 b u^3 + \frac{5}{18}\alpha(5b-4) u_{xx} + \frac{5}{2}\alpha(2-3b) u^2 + \tilde{K}_2 u + \tilde{K}_1 = 0, \quad K_1, \tilde{K}_1, \tilde{K}_2 = \text{const}. \tag{15}$$