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# Analytical results on the magnetization of the Hamiltonian Mean-Field model

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ABSTRACT

The violent relaxation and the metastable states of the Hamiltonian Mean-Field model, a paradigmatic system of long-range interactions, is studied using a Hamiltonian formalism. Rigorous results are derived algebraically for the time evolution of selected macroscopic observables, e.g., the global magnetization. The high- and low-energy limits are investigated and the analytical predictions are compared with direct $N$-body simulations. The method we use enables us to re-interpret the out-of-equilibrium phase transition separating magnetized and (almost) unmagnetized regimes.



## 1. Introduction

Systems with long-range interactions [1,2] exhibit a fascinating feature of metastability: Starting from out-of-equilibrium initial conditions, the system violently relaxes toward a metastable state, often called Quasi-Stationary State (QSS). In this regime, macroscopic quantities reach values which substantially differ from the corresponding thermodynamic equilibrium configuration. Although the QSS are only transient regimes, their lifetime have been shown to diverge with the number of bodies in interaction [3]. For this reason they possibly correspond to the solely accessible experimental regimes.

We consider a paradigmatic system with long-range interactions, the Hamiltonian Mean-Field (HMF) [3] where particles on a circle are collectively interacting through a cosine-like mean-field potential. After a fast relaxation, the system typically enters a metastable regime in which the particles either aggregate into a large cluster (magnetized phase), or they spread almost homogeneously around the circle (unmagnetized or homogeneous phase). In particular, an out-of-equilibrium phase transition between these two states occurs when the parameters of the initial conditions are varied [4].

In this Letter, we focus on both the violent relaxation process and the subsequent QSS regime. We use an algebraic framework based on a Hamiltonian formulation of the Vlasov equation for the HMF model. This Vlasov equation rules the evolution of the single particle distribution function in phase space (as a kinetic equation) and naturally arises when investigating the continuous version of the HMF model. As in the limit of infinite number of particles the system gets permanently frozen in the QSS phase, it is customarily believed that QSS can be interpreted as equilibria of the Vlasov equation. We exploit a Hamiltonian formalism of this Vlasov equation to derive analytical expressions for the global magnetization as function of time. This magnetization measures the aggregation of the particles on the circle. It is a macroscopic observable which is directly influenced by the microscopic, single particle trajectory. It is in general particularly cumbersome to bridge the gap between the microscopic realm of the many-body interacting constituents and the macroscopic world of collective dynamics.

Using an expansion provided by the Hamiltonian framework, we here obtain rigorous results on the time expansion of relevant observables. These results are compared with direct numerical simulation. We consider in particular the high- and low-energy regimes which allow some simplifications in the expansions. In

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addition, we characterize the aforementioned out-of-equilibrium phase transition which occurs in an intermediate energy range. This is achieved by monitoring the initial relaxation of the magnetization, as a function of relevant parameters of the initial distribution. The parameter space is hence partitioned into two regions, depending on the magnetization amount, a result which positively correlates with direct numerics [4].

The Letter is organized as follows: in Section 2 we will review the discrete HMF model, present its continuous counterpart and discuss the basic of the bracket expansion method. Section 3 is devoted to the presentation of the analytical results, with special emphasis to the high- and low-energy regimes. The out-of-equilibrium phase transition issue is also addressed. Comparison with direct simulations is provided to substantiate the accuracy of our predictions.

## 2. Model and methods

### 2.1. Lie–Poisson structure of the Vlasov equation

We consider $N$ particles interacting on a circle with the following Hamiltonian:

$$H=\sum_{i=1}^{N}\left[\frac{p_i^2}{2}+\frac{1}{2N}\sum_{j=1}^{N}\left[1-\cos(\theta_i-\theta_j)\right]\right], \tag{1}$$

where $(\theta_i, p_i)$ are canonically conjugate variables which means that the Poisson bracket giving the dynamics (Hamilton's equations) is given by

$$\{F,G\}=\sum_{i=1}^{N}\left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial \theta_i}-\frac{\partial F}{\partial \theta_i}\frac{\partial G}{\partial p_i}\right).$$

In the continuous limit, we consider an Eulerian description of the system which gives the dynamical evolution of the distribution of particles $f(\theta,p;t)$ in phase space via the following Vlasov equation:

$$\frac{\partial f}{\partial t}=-p\frac{\partial f}{\partial \theta}+\frac{dV[f]}{d\theta}\frac{\partial f}{\partial p}, \tag{2}$$

where $V[f](\theta)=1-M_x[f]\cos\theta-M_y[f]\sin\theta$. The magnetization $M[f]=M_x+iM_y$ is defined as

$$M[f]=\iint fe^{i\theta}\,d\theta\,dp, \tag{3}$$

where the integrals are taken over $[-\pi,\pi]\times\mathbb{R}$. Eq. (2) can be cast into a Hamiltonian form where the (infinite-dimensional) phase space is composed of the functions $f(\theta,p)$ of $]-\pi,\pi]\times\mathbb{R}$. The Hamiltonian is given by

$$H[f]=\iint f\frac{p^2}{2}\,d\theta\,dp-\frac{M_x[f]^2+M_y[f]^2-1}{2}, \tag{4}$$

and the associated Lie–Poisson bracket by

$$\{F,G\}=\iint f\left(\frac{\partial}{\partial p}\frac{\delta F}{\delta f}\frac{\partial}{\partial \theta}\frac{\delta G}{\delta f}-\frac{\partial}{\partial \theta}\frac{\delta F}{\delta f}\frac{\partial}{\partial p}\frac{\delta G}{\delta f}\right)d\theta\,dp, \tag{5}$$

for $F$ and $G$ two observables (that is, functionals of $f$). The functional derivatives $\delta F/\delta f$ are computed following the expansion:

$$F[f+\varphi]-F[f]=\iint\frac{\delta F}{\delta f}\varphi\,d\theta\,dp+O\left(\varphi^2\right).$$

The Poisson bracket (5) satisfies several properties: bilinearity, Leibniz rule and Jacobi identity (for more details, see Refs. [5,6]). Its Casimir invariants are given by

$$C[f]=\iint c(f)\,d\theta\,dp,$$

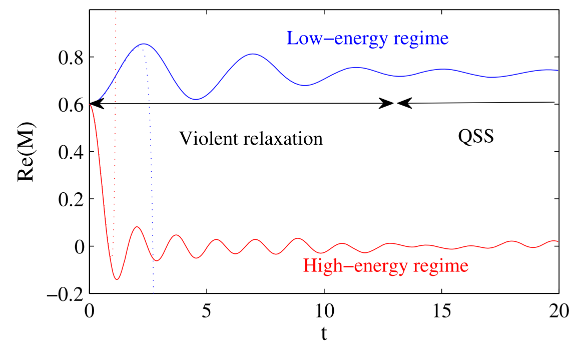


**Fig. 1.** Real part of the magnetization given by Eq. (3) as a function of time obtained by integrating the dynamics given by Eq. (6) for $M_0=0.6$. The system reaches either a finite-magnetization for low energies ($U=0.4$, in blue), or a low-magnetization for high energies ($U=3$, in red). The plain lines refer to $N$-body simulations (with $N=10\,000$), while the dotted lines come from the predictions given by Eq. (9) for $k_0=20$. (For interpretation of colors in this figure, the reader is referred to the web version of this article.)

where $c(f)$ is any function of $f(\theta,p)$. In particular, the total distribution $\iint f\,d\theta\,dp$ is one of such Casimir invariants and hence is conserved by the flow. The evolution of any observable $F[f]$ is then given by

$$\dot{F}=\{H,F\}. \tag{6}$$

For instance, for $F[f]=f(\theta,p)$, we recover Eq. (2). Another convenient observable to study is the magnetization $M[f]$ given by Eq. (3): It quantifies the spatial aggregation of the particles. At low energies, the magnetization typically relaxes until it reaches an out-of-equilibrium plateau, around which it fluctuates (see Fig. 1). In this case, the particles are trapped into the large resonance created by the finite magnetization, hence the name "magnetized state" (see upper panel of Fig. 2). At high energies, the magnetization falls and fluctuates around zero (see Fig. 1), which means that the particles failed to organize collectively. This is called the "homogeneous phase" (see lower panel of Fig. 2).

The dynamics given by Eq. (6) is deduced from the linear operator $\mathcal{H}$. From the evaluation of the functional derivative of $H$ with respect to $f$

$$\frac{\delta H}{\delta f}=\frac{p^2}{2}-M_x[f]\cos\theta-M_y[f]\sin\theta,$$

we get the expression of $\mathcal{H}$:

$$\mathcal{H}\equiv\{H,.\}$$
$$=\iint d\theta\,dp\,f\left(p\frac{\partial}{\partial\theta}+\frac{Me^{-i\theta}-M^*e^{i\theta}}{2i}\frac{\partial}{\partial p}\right)\frac{\delta}{\delta f}. \tag{7}$$

In the algebraic computations that follow, we make an explicit use of the linearity of $\mathcal{H}$ and Leibniz rule:

$$\mathcal{H}(F+\alpha G)=\mathcal{H}F+\alpha\mathcal{H}G,$$
$$\mathcal{H}(FG)=F\mathcal{H}G+(\mathcal{H}F)G.$$

*$N$-body simulations* (*Lagrangian point of view*): In order to compare the algebraic results with numerical ones, we integrate Eq. (6) via $N$-body simulations, which are obtained by considering a Klimontovitch [7] distribution of particles

$$f(\theta,p;t)=\frac{1}{N}\sum_{i=1}^{N}\delta\big(\theta-\theta_i(t)\big)\delta\big(p-p_i(t)\big),$$

whose dynamics is ultimately reduced to Hamiltonian (1). Such simulations are used with a large number of particles (typically $N=10^5$) such that the intermediate regime experienced by the