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Coevolutionary extremal dynamics on gasket fractal

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ABSTRACT

We considered a Bak–Sneppen model on a Sierpinski gasket fractal. We calculated the avalanche size distribution and the distribution of distances between subsequent minimal sites. To observe the temporal correlations of the avalanche, we estimated the return time distribution, the first-return time, and the all-return time distribution. The avalanche size distribution follows the power law, $P(s) \sim s^{-\tau}$, with the exponent $\tau = 1.004(7)$. The distribution of jumping sites also follows the power law, $P(r) \sim r^{-\pi}$, with the critical exponent $\pi = 4.12(4)$. We observe the periodic oscillation of the distribution of the jumping distances which originated from the jumps of the level when the minimal site crosses the stage of the fractal. The first-return time distribution shows the power law, $P_f(t) \sim t^{-\tau_f}$, with the critical exponent $\tau_f = 1.418(7)$. The all-return time distribution is also characterized by the power law, $P_a(t) \sim t^{-\tau_a}$, with the exponent $\tau_a = 0.522(4)$. The exponents of the return time satisfy the scaling relation $\tau_f + \tau_a = 2$ for $\tau_f \leqslant 2$.

1. Introduction

Many complex systems show self-organized criticality (SOC) without the control parameters [1–5]. The Bak–Sneppen (BS) model originated from the macroscopic coevolutionary dynamics of biological systems. The evolution takes place intermittently. After a long calm state, the speciation or extinction occurs abruptly. The BS model is a simple system showing SOC and follows extremal dynamics. In extremal dynamics, a site with some globally extremal value is updated and dynamically interacts with nearby sites. The system evolves toward the critical state without any characteristic scales. Many systems showing the SOC have been introduced; for example, an avalanche of sand or rice, invasion percolation for fluid displacement in porous media, the Sneppen model for the dynamics of surfaces pinned by quenched disorder, and the Olami–Feder–Christensen model which describes the motion of tectonic plates, etc. [6–10].

In the BS model, each species on the lattice with size N has fitness value f_i which is generated from the uniform random number [5,10]. The species with the lowest fitness is updated by new randomly-selected fitness. Simultaneously, the fitness of its nearest neighbors is also updated. Only the species with minimal fitness and her nearest neighbors become extinct and are replaced by new species. As this process is repeated, the system approaches the critical states. At the critical steady-state, the fitness of the species is greater than the critical threshold f_c. In the BS model, the active site is defined by the site having the fitness with $f_i \leqslant f_c$. When a minimal fitness site becomes active, this activity propagates like an avalanche. This avalanche stops when all the fitness levels of the sites are greater than f_c again. The duration time s of an avalanche is called by an avalanche size. The probability distribution of the avalanche size follows the power law. The criticalities of the BS model were reported on the regular lattice, small-world networks, and the scale-free network [11–24]. The universality of the BS model has been studied for the lattice systems [10]. We summarized the critical exponents of the BS model in Table 1 for the isotropic cases. The critical exponents depend on the dimensionality of the lattice and also on the symmetry. For the anisotropic BS model, the critical exponents showed different values in comparison to the isotropic BS model [16,17]. The upper critical dimension of the BS model is still controversial [11,12]. Boettcher and Paczuski proposed $d_c = 4$ [12], but De Los Rios et al. reported $d_c = 8$ [11]. At the $d > d_c$ the critical exponents followed the mean-field exponents. The exponent τ of the avalanche size distribution is $\tau = 3/2$ at the mean-field [10].

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Table 1
The critical exponents of the isotropic Bak–Sneppen model. The exponents are from Ref. [10] for 1-D and 2-D lattices and from Ref. [11,12] for 3-D and 4-D lattices.

Lattice	τ	π	τ_f	τ_a
1D lattice	0.914(4)	3.23(2)	1.58(1)	0.45(2)
SG	1.004(7)	4.12(4)	1.418(7)	0.522(4)
2D lattice	1.25(1)	–	1.28(3)	0.70(3)
3D lattice	1.35	–	1.09	0.92
4D lattice	1.41	–	1.16	1.15

The lattice and network structure influence to the critical behaviors of the SOC models. With investigating the sandpile model on the two-dimensional Sierpinski gasket fractal, Kutnjak-Urbanc et al. validated the existence of the SOC phenomena and critical scaling properties on deterministic fractal unlike the Ising model below two dimensions [28]. Vanderzande and Daerden predicted a scaling relation, $\nu = 1/d_f$, where ν is the critical exponent of the correlation length and d_f is the fractal dimension of the dissipated Abelian sandpile model on two-dimensional Sierpinski gasket fractal [29]. The fractal structures have been found in many complex ecological system [30–35]. Therefore, it is interesting to study the Bak–Sneppen model of the biological evolution on a fractal structure. The critical exponents of the BS model on the fractal are non-trivial. To our knowledge, the BS model have not been reported on a fractal structure until now. In this article we consider the Bak–Sneppen model on the Sierpinski gasket fractal [25–27]. We investigate the effect of the dimensionality and the topology on the critical behavior of the BS model. We confirm the scaling properties and the critical exponents of the BS model on the Sierpinski gasket as their analogy for the sand pile models [28, 29]. The fractal structure is characterized by the fractal dimension d_f. By self-similarity, the fractal dimension is not the integer but the real number. The Sierpinski gasket fractal has the fractal dimension $d_f = \ln 3/\ln 2 = 1.58496\ldots$. We observe that the critical exponents are changed by the fractal structure. However, the scaling relations hold when we replace the dimension to the fractal dimension. The critical exponents of the first and all-return time distribution satisfied the scaling relation for the gasket fractal. We introduce the gasket fractal and Monte Carlo method in Section 2. We present the results and discussions in Section 3. We summarize results in Section 4.

2. Gasket fractal and Monte Carlo method

We generated the Sierpinski gasket fractal on two-dimensional Euclidean space. The generator of the Sierpinski gasket was an equilateral triangle. In the first stage $n = 1$, we cut out the inverse triangle that is formed by connecting the middle points of each side. We repeated this procedure for the remaining upright triangles. At the stage n, the total number of the lattice point was equal to $N(n) = 3(3^n + 1)/2$. At the stage $n = 8$, the total number of lattice sites was $N(8) = 9843$. We generated the gasket fractal at the stage $n = 6, 7, 8$. On the gasket fractal, we simulated the Bak–Sneppen model. We assigned the fitness values on the each lattice site of the gasket fractal. At each Monte Carlo step we updated the fitness of the minimal fitness site and its nearest neighbors sites. We then repeated these processes. At steady-state, the critical fitness was $f_c = 0.4069$ which is smaller than the one-dimensional critical fitness $f_c = 0.66736$ [26]. We observed the distribution of the avalanche size and the return-time distribution at the steady-state.

3. Results and discussions

Fig. 1 shows the normalized probability distribution $P(s)$ of the avalanche size as a function of the duration time s. The distribution of the avalanche size shows the power law,

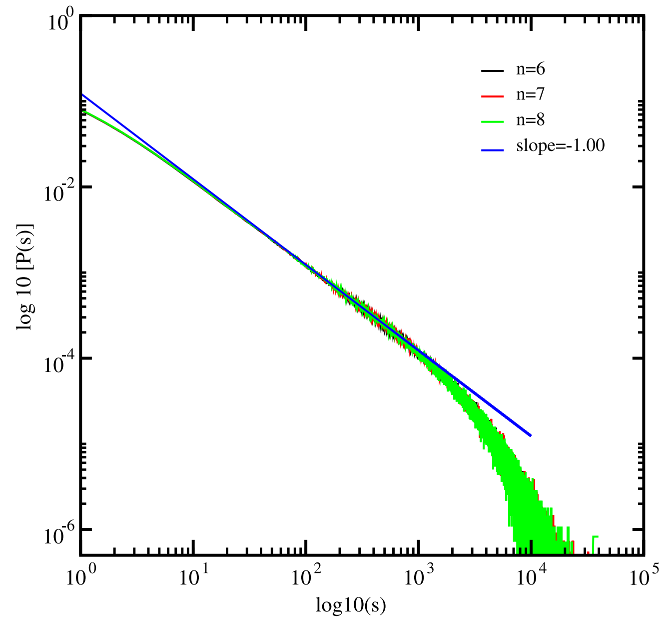

Fig. 1. The probability distribution of the avalanche size for the different size of the gasket fractal at the stage $n = 6, 7, 8$. The normalized probability distribution shows a power law, $P(s) \sim s^{-\tau}$. We obtained the critical exponent $\tau = 1.004(7)$ by a least squares fit. (To distinguish between different color lines in the figures, the reader is referred to the web version of this article.)

$$P(s) \sim s^{-\tau}. \tag{1}$$

We obtained the critical exponent $\tau = 1.004(7)$ by a least squares fit. The exponent τ is located between the value $\tau(1D) = 0.914$ of the one-dimensional lattice and the value $\tau(2D) = 1.25$ of the two-dimensional lattice. The critical exponents are summarized in Table 1 and compared with previous results of the lattice. We also estimated the covered sites $V(s)$ of an avalanche as a function of the duration time s. The average covered sites followed the power law, $V(s) \sim s^{\mu}$. The exponents τ and μ are a set of the basic exponents for the universality of the Bak–Sneppen model. We obtained the exponent $\mu = 0.58(7)$ for the gasket fractal.

Fig. 2 shows the normalized probability distribution $P(r)$ of the jumping distances between subsequent minimal sites as a function of the distance r. When an avalanche propagates on the gasket fractal, an active site and its nearest sites are updated simultaneously. Then, the next minimal site is a distance r away. The distribution $P(r)$ follows the power law,

$$P(r) \sim r^{-\pi}, \tag{2}$$

with periodic modulation. We obtained the exponent, $\pi = 4.12(4)$ for the gasket fractal. We observed the periodic local peaks of the distribution. These kinds of oscillations have been observed for the distribution functions in different systems on the Sierpinski gasket fractal [25,28]. The peaks originated from the level jump of the gasket fractal. When the site of the minimal fitness jumps from the stage $n+1$ to the stage n on the gasket fractal, the Euclidean distance also jumps by the discrete scale invariant nature of the underlying fractal lattice. The exponent π for the gasket fractal is larger than the exponent $\pi = 3.19$ for the one-dimensional lattice.

The critical exponents are expressed by the base exponents (τ, μ). The covered sites of an avalanche are fractal structure, $V(s) = R(s)^D$ where $R(s)$ is the radius of gyration of the fractal. The exponent μ is related to the exponent D as $\mu = d/D$ where d is the spatial dimension of the lattice [10]. In the Sierpinski gasket d is the fractal dimension $d = d_f$. The exponent π satisfies a scaling relation $\pi = 1 + D(2 - \tau)$. From this relation we obtained the exponent $D = (\pi - 1)/(2 - \tau)$. Then, we obtained the expo-