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Signal-to-noise ratio improvement by stochastic resonance in moments in non-dynamical systems with multiple states

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ABSTRACT

Conventional stochastic resonance in terms of signal-to-noise ratio refers to the amplification of a weak signal in the average of the output. In this framework, only the first moment of the output is used for extracting the information about the input signal. However, higher order moments are also modulated by the input signal. We report the occurrence of stochastic resonance in higher moments. Furthermore, by the linear combination of moments, the signal-to-noise ratio improves compared with the conventional method, which uses the lowest moment only.

1. Introduction

Noise has been regarded as a cause of deterioration of the signal processing performance. However, about three decades ago, Benzi et al. observed the enhancement of a weak signal buried in noise by tuning the noise intensity [1,2]. This phenomenon is called *stochastic resonance* (SR). Since the discovery of SR, the positive application of noise has been widely explored for signal processing [3,4]. SR is found in a variety of systems such as the Schmitt trigger [5], neuron models [6], and superconducting quantum interference devices (SQUIDs) [7]. For these systems, intensive investigations have been carried out for signal processing such as amplification of weak signals by the SR effect.

The present Letter discusses the signal processing performance that is achieved by the use of SR. *Signal-to-noise ratio* (SNR) is one of the major performance metrics in signal processing field. Even in a recent signal detection scheme, keeping high SNR is a key factor [8,9]. From the other aspects of sensing, for example, localization with radio signaling also depends on the SNR [9]. For these reasons, many studies on SR try to find a method to achieve the high SNR.

In practical applications, the nonlinearity of a device is often fixed, and only the output from the device is observed. Then our objective is how to detect the input signal from the observed output. Insightful readers may answer the Bayesian scheme gives the best way. However, because of its computing cost and time, a number of devices are not designed on the basis of Bayesian. We propose the easily applicable method to improve the SNR without any extra devices. The goal of this Letter is to improve the SNR in discrete systems, since digital systems are broadly used in modern signal processing. In such systems such as analog-to-digital converters, the nonlinearity is inevitable.

What is conventionally of concern with SR in terms of SNR is the average of the output, namely, the first moment of the output only. In the literature [10–12], the SR effect only of the first moment has been investigated, even for a system with multiple discrete states. In such a system, the so called empirical probability

$$P_N(a) \equiv \frac{1}{N}\sum_{n=0}^{N-1}\delta_{a,x_n} \tag{1}$$

is influenced by the input signal. Here x_n denotes the n-th sample of the output, a is a discretized state of the output, and N is the number of sampled output. Equivalently, the k-th empirical moment

$$M_k \equiv \frac{1}{N}\sum_{n=0}^{N-1}x_n^k = \sum_{l=0}^{K-1}a_l^k P_N(a_l) \tag{2}$$

is modulated, and utilized to extract the information on the input signal. Here a_k is the k-th state of the output and K is the number of states of the output. The empirical probabilities $P_N(a_0),\dots,P_N(a_{K-1})$ are determined by the K moments $M_0 = 1,\dots,M_{K-1}$. Since each moment ($k<K$) is linearly independent

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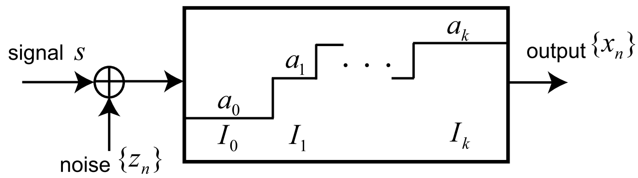

Fig. 1. Schematic system-flow diagram of a system with discrete states.

of each other, it contains different information on the input signal. In this sense, the conventional SR is insufficient for achieving the high SNR. Note that the output moments are not simply related to each other even for Gaussian noise, since discretization is a nonlinear procedure. We focus on the SNR achieved by exploiting the higher order moments. The contributions of this Letter are: (i) the discovery of the occurrence of SR in higher moments of the output, and (ii) the proposal of the method for the SNR improvement by the appropriate linear combination of the empirical moments.

The remainder of the present Letter is organized as follows. In the next section, we report the occurrence of SR in the higher order moments. In Section 3, we show that the appropriate linear combination of empirical moments yields the SNR higher than that for any single moment. The combined moments can also exhibit SR behavior. The last section is devoted to the conclusion of the present Letter.

2. SR in moments of non-dynamical system with discrete states

We investigate the SNR of the moments of the output from the system with K discrete states [10–12] described by

$$x_n = F(s_n + z_n), \tag{3}$$

where s_n, z_n, and x_n represent an input signal, noise, and the output, respectively, of the n-th sample. Throughout the present Letter, we deal with a white noise, i.e., $\langle z_n z_m\rangle \propto \delta_{nm}$, where $\langle\cdot\rangle$ denotes the expectation value with respect to the noise $\{z_n\}$. The nonlinear function F are assumed to be known and fixed, and the input noise follows a stationary probability density. Furthermore, we assume that the input signal s_n is small compared to the noise intensity, since a weak signal is of our interest. The function F denotes the output of K discrete states: $F(X) = a_k$ for $X \in I_k$ $(k = 0, 1, \dots, K-1)$, where the domains $\{I_k\}$ satisfy $I_k \cap I_l = \phi$ for $k \neq l$ and $\bigcup_{k=0}^{K-1} I_k = \mathbb{R}$. The real constants $\{a_k\}$ satisfy $a_k \neq a_l$ for $k \neq l$. We assume that the number of samples N satisfies $N \geqslant K-1$ in order to obtain $K-1$ linearly independent moments. A schematic system-flow diagram is described in Fig. 1.

We are particularly interested in a periodic input signal. Since the signal is assumed to be small, we can apply the linear response theory, and the output for any input signal is described as the superposition of different Fourier components. For a periodic input, as in the case of standard Fourier transform, the output signals are sampled periodically. Because of the periodicity of the input, we focus on the constant signal $s_n = s$ hereinafter.

In order to define SNR of moments, we identify the signal and noise parts of the output moments as follows. Since the input signal $s_n = s$ is assumed to be sufficiently small compared with the noise intensity, the empirical moment M_k can be described in the framework of linear response theory as

$$M_k = m_k + \chi_k s + \eta_k + O\left(s^2\right), \tag{4}$$

where m_k is the expectation value of the k-th order moment without the input s, i.e., $m_k = \langle F^k(z)\rangle$, χ_k is the response of the k-th order moment $\chi_k = \langle \partial F^k(z)/\partial z\rangle$, and η_k denotes the fluctuation of

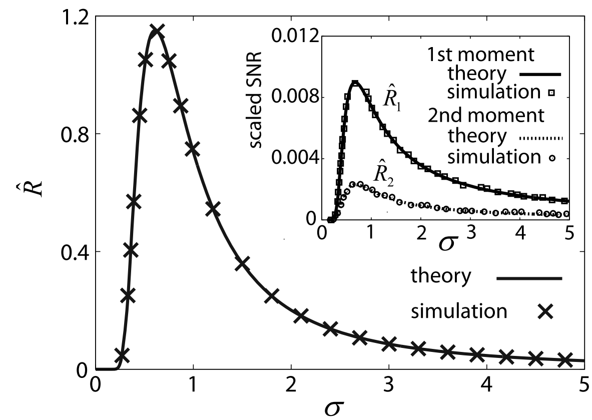

Fig. 2. Scaled SNR for the linear combination of moments $\hat{R}$ and for the first and the second moments $\hat{R}_1$ and $\hat{R}_2$ (inset), in the presence of white Gaussian noise. The theoretical results (solid and dashed lines) are compared with the simulation results (cross marks, circles and squares).

the k-th order moment, which is not influenced by the input signal. In this expression, $\chi_k s$ represents the deterministic part, i.e., signal part in the empirical moment, and η_k corresponds to the noise part. Since all moments are linear in the signal amplitude, the SNR for the k-th order moment M_k is given by

$$R_k = s^2 N \hat{R}_k, \qquad \hat{R}_k = \chi_k^2 / V_{kk},$$
$$V_{kl} \equiv \left\langle \left[F^k(z) - m_k\right]\left[F^l(z) - m_l\right]\right\rangle. \tag{5}$$

In this Letter, $\hat{R}_k$ is referred as the *scaled SNR* for the k-th order moment. Note that the SNR is quadratic in the signal amplitude, and it is fully determined by the first moment of the signal.

If the transfer function F and the probability density of the noise z_n are both known, the offset m_k, the response function χ_k, and the covariance of the output V_{kl} all can be calculated. Then the scaled SNR is theoretically calculated. We can evaluate the value of the input signal s by the ratio of the observed SNR R_k and the scaled SNR $\hat{R}_k$.

In Fig. 2, the scaled SNR $\hat{R}_k$ is plotted as a function of the standard deviation σ of the noise z_n. A white Gaussian noise with zero mean, i.e., $\langle z_n\rangle = 0$, $\langle z_n z_m\rangle = \sigma^2\delta_{nm}$, has been used. In the simulation, the input signal is sinusoidal wave with amplitude 0.1 and frequency 0.001. The function F has been taken as a 3-valued function: $F(X) = a_0$ for $X < \theta_1$, $F(X) = a_1$ for $\theta_1 \leqslant X < \theta_2$, and $F(X) = a_2$ for $X \geqslant \theta_2$ with the parameter sets $(a_0, a_1, a_2) = (3.0, -10.0, 1.0)$ and $(\theta_1, \theta_2) = (-1.0, 1.0)$, i.e., $I_0 = (-\infty, -1.0)$, $I_1 = [-1.0, 1.0)$, and $I_2 = [1.0, \infty)$ (see Fig. 3). The 3-valued system is the simplest system that exhibits nontrivial results. The simulation results are the average over 160 periods of the input signal, and 6000 samples are taken in a period. The theoretical curves are compared with numerical simulations, which provide an independent test of the theory.

According to Fig. 2, the SNR for the first and the second moments both exhibit resonance-like peaks that imply the occurrence of SR. Generally, the locations and the heights of the peaks depend on the choice of $\{a_k\}$ and $\{I_k\}$. Moreover, the occurrence of the SR depends on the choice of $\{I_k\}$. According to the definition of SR, which refers to the peak of SNR as a function of noise intensity, the thresholds of a system exhibiting SR are ill-tuned in the presence of weak noise. If one of the thresholds is set to be zero, the system does not exhibit a resonance-like peak in SNR, and the output SNR becomes monotonically decreasing function with respect to the input noise intensity.