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Analytical expressions of global quantum discord for two classes of multi-qubit states

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ABSTRACT

Global quantum discord (GQD), proposed by Rulli and Sarandy [C.C. Rulli, M.S. Sarandy, Phys. Rev. A 84 (2011) 042109], is a generalization of quantum discord to multipartite states. In this Letter, we provide an equivalent expression for GQD, and obtain the analytical expressions of GQD for two classes of multi-qubit states. The phenomena of sudden transition and freeze of GQD are also discussed.

1. Introduction

Quantifying the multipartite quantum correlations is a very challenging and still largely open question [1–5]. For bipartite case, entanglement and quantum discord have been widely accepted as two fundamental tools to quantify quantum correlations [6,7], and quantum discord captures more quantum correlations than entanglement in the sense that a separable state may have nonzero quantum discord. Generalizations of bipartite quantum discord to multipartite states have been considered in different ways [8–11]. In [11], Rulli and Sarandy proposed a measure for multipartite quantum correlations, called global quantum discord (GQD), which can be seen as a generalization of bipartite quantum discord [12,13] to multipartite states. GQD is always nonnegative and its use is illustrated by the Werner-GHZ state and the Ashkin–Teller model [11].

In this Letter, we provide an equivalent expression for GQD (Section 3), and derive the analytical expressions of GQD for two classes of multi-qubit states (Section 4), these results generalize the earlier results [12,14]. The phenomena of sudden transition and freeze of GQD are also discussed (Section 5). For clarity of reading, we first recall the definition of GQD proposed in [11] (Section 2).

2. Global quantum discord (GQD)

We briefly review the definition of GQD proposed in [11].

Consider two systems A_1 and A_2 (each of them is finite dimensional), the symmetric quantum discord of a state $\rho_{A_1A_2}$ of the composite systems A_1A_2 is

$$D(\rho_{A_1A_2})=\min_{\Phi}\left[I(\rho_{A_1A_2})-I\left(\Phi_{A_1A_2}(\rho_{A_1A_2})\right)\right]. \tag{1}$$

In Eq. (1),

$$I(\rho_{A_1A_2})=S(\rho_{A_1})+S(\rho_{A_2})-S(\rho_{A_1A_2}), \tag{2}$$

is the mutual information of $\rho_{A_1A_2}$, min is taken over all locally projective measurements performing on AB, $\Phi_{(\cdot)}$ denotes a locally projective measurement performing on the system $(\cdot)$, $S(\cdot)$ is the Von Neumann entropy, and ρ_{A_1}, ρ_{A_2} are reduced states of $\rho_{A_1A_2}$.

$D(\rho_{A_1A_2})$ is a natural extension of the original definition of quantum discord which defined over all projective measurements performing only on A_1 or A_2 [12,13].

Since the mutual information $I(\rho_{A_1A_2})$ can be expressed by the relative entropy

$$I(\rho_{A_1A_2})=S(\rho_{A_1A_2}\|\rho_{A_1}\otimes\rho_{A_2}), \tag{3}$$

hence, Eq. (1) can also be recasted as

$$D(\rho_{A_1A_2})=\min_{\Phi}\left[S(\rho_{A_1A_2}\|\rho_{A_1}\otimes\rho_{A_2}) - S\left(\Phi_{A_1A_2}(\rho_{A_1A_2})\|\Phi_{A_1}(\rho_{A_1})\otimes\Phi_{A_2}(\rho_{A_2})\right)\right]. \tag{4}$$

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Note that the relative entropy of state ρ with respect to state σ (ρ and σ lie on the same Hilbert space) is defined as

$$S(\rho\|\sigma) = tr(\rho\log_2\rho) - tr(\rho\log_2\sigma). \tag{5}$$

It is easy to check that Eq. (1) can be rewritten as

$$D(\rho_{A_1A_2}) = \min_{\Phi}\big[S\big(\rho_{A_1A_2}\|\Phi_{A_1A_2}(\rho_{A_1A_2})\big) - S\big(\rho_{A_1}\|\Phi_{A_1}(\rho_{A_1})\big) - S\big(\rho_{A_2}\|\Phi_{A_2}(\rho_{A_2})\big)\big]. \tag{6}$$

The definition of GQD is a generalization of bipartite symmetric quantum discord. Consider N ($2 \leqslant N < \infty$) systems $A_1, A_2, \dots, A_N$ (each of them is finite dimensional), the GQD of state $\rho_{A_1A_2\dots A_N}$ on the composite system $A_1A_2\dots A_N$ is defined as [11]

$$D(\rho_{A_1A_2\dots A_N}) = \min_{\Phi}\left[S\big(\rho_{A_1A_2\dots A_N}\|\Phi_{A_1A_2\dots A_N}(\rho_{A_1A_2\dots A_N})\big) - \sum_{j=1}^{N} S\big(\rho_{A_j}\|\Phi_{A_j}(\rho_{A_j})\big)\right]. \tag{7}$$

It has been proved that $D(\rho_{A_1A_2\dots A_N}) \geqslant 0$ for any state $\rho_{A_1A_2\dots A_N}$ [11]. Also, it is easy to see that $D(\rho_{A_1A_2\dots A_N})$ keeps invariant under any locally unitary transformation.

3. An equivalent expression for global quantum discord

In this section, we provide an equivalent expression for GQD, from this equivalent expression we can get an interpretation for GQD. We first state two mathematical facts as the lemmas below, which are simple but useful.

Lemma 1. *For any square matrix (with finite dimensions) A, let $\overline{A}$ be the matrix whose diagonal elements are the same with A, and other elements are zero. B and $\overline{B}$ are defined similarly. Then*

$$tr(A\overline{B}) = tr(\overline{A}\,\overline{B}), \tag{8}$$

$$tr\big(Af(\overline{B})\big) = tr\big(\overline{A}f(\overline{B})\big), \tag{9}$$

where $f(\,)$ is any function.

Lemma 2. *Let $\rho_{A_1A_2\dots A_N}$ be a state on Hilbert space $H_{12\dots N}$, ρ_{A_1}, $\rho_{A_2}, \dots, \rho_{A_N}$ be the reduced states of $\rho_{A_1A_2\dots A_N}$ on Hilbert spaces $H_1, H_2, \dots, H_N$, respectively. Suppose $\sigma_{A_1}, \sigma_{A_2}, \dots, \sigma_{A_N}$ are states on $H_1, H_2, \dots, H_N$, respectively. Then it holds that*

$$tr\big[\rho_{A_1A_2\dots A_N}\log_2(\sigma_{A_1}\otimes\sigma_{A_2}\otimes\dots\otimes\sigma_{A_N})\big] = \sum_{i=1}^{N} tr_i[\rho_{A_i}\log_2\sigma_{A_i}]. \tag{10}$$

Proof. We only prove the case of $N = 2$, the proof of $N > 2$ is similar. When $N = 2$, we need to prove

$$tr\big[\rho_{A_1A_2}\log_2(\sigma_{A_1}\otimes\sigma_{A_2})\big] = tr_1[\rho_{A_1}\log_2\sigma_{A_1}] + tr_2[\rho_{A_2}\log_2\sigma_{A_2}]. \tag{11}$$

It is known that $\rho_{A_1A_2}$ can be written as [15]

$$\rho_{A_1A_2} = \sum_j c_j\rho_{1j}\otimes\rho_{2j}, \tag{12}$$

where $\{c_j\}_j$ are real numbers, ρ_{1j}, ρ_{2j} are all Hermitian matrices. For σ_{A_1}, σ_{A_2}, there exist unitary matrices U_1 and U_2 such that $D_1 = U_1\sigma_{A_1}U_1^+$, $D_2 = U_2\sigma_{A_2}U_2^+$ are all diagonal, where $+$ denotes adjoint. Note that

$$\log_2(D_1\otimes D_2) = (\log_2 D_1)\otimes I_2 + I_1\otimes(\log_2 D_2), \tag{13}$$

where I_1, I_2 are the identity operators on H_1, H_2, respectively. Then Eq. (11) can be directly verified. □

With the help of Lemma 1 and Lemma 2, we can get an equivalent expression for GQD defined by Eq. (7).

Theorem 1. *The GQD of a state $\rho_{A_1A_2\dots A_N}$ defined by Eq. (7) can also be expressed as*

$$D(\rho_{A_1A_2\dots A_N}) = \min_{\Phi}\big[I(\rho_{A_1A_2\dots A_N}) - I\big(\Phi_{A_1A_2\dots A_N}(\rho_{A_1A_2\dots A_N})\big)\big], \tag{14}$$

where, the mutual information

$$I(\rho_{A_1A_2\dots A_N}) = \sum_{i=1}^{N} S(\rho_{A_i}) - S(\rho_{A_1A_2\dots A_N}). \tag{15}$$

Proof. From Lemma 1 and Lemma 2, we have

$$S\big(\rho_{A_j}\|\Phi_{A_j}(\rho_{A_j})\big) = -S(\rho_{A_j}) + S\big(\Phi_{A_j}(\rho_{A_j})\big), \tag{16}$$

$$I(\rho_{A_1A_2\dots A_N}) = \sum_{i=1}^{N} S(\rho_{A_i}) - S(\rho_{A_1A_2\dots A_N}) = S(\rho_{A_1A_2\dots A_N}\|\rho_{A_1}\otimes\rho_{A_2}\otimes\dots\otimes\rho_{A_N}). \tag{17}$$

Then we can easily prove Theorem 1. □

From Theorem 1, we see that, GQD of a state is just the minimal loss of mutual information over all locally projective measurements. This interpretation of GQD is consistent with the symmetric quantum discord in Eq. (1), as well as the original definition of quantum discord for bipartite states.

We remark that, Eq. (14) is also investigated in [9], and a witness for nonzero-GQD states was proposed in [16].

For a special case, we consider a state $\rho_{A_1A_2\dots A_N}$ whose reduced states $\rho_{A_1}, \rho_{A_2}, \dots, \rho_{A_N}$ are all proportional to identity operator. In such case, the GQD of $\rho_{A_1A_2\dots A_N}$ can be remarkably simplified. We state it as Theorem 2, which can be directly obtained from Eq. (7) or from Eq. (14).

Theorem 2. *An N-partite state $\rho_{A_1A_2\dots A_N}$, if its reduced states $\rho_{A_1}, \rho_{A_2}, \dots, \rho_{A_N}$ are all proportional to identity operator, then the GQD of $\rho_{A_1A_2\dots A_N}$ can be expressed as*

$$D(\rho_{A_1A_2\dots A_N}) = -S(\rho_{A_1A_2\dots A_N}) + \min_{\Phi} S\big(\Phi_{A_1A_2\dots A_N}(\rho_{A_1A_2\dots A_N})\big). \tag{18}$$

4. Two classes of N-qubit states which allow analytical expressions of GQD

Recall that, whichever the original quantum discord [17], the geometric discord [18], the two-sided quantum discord [14], and the two-sided geometric discord [19], they are all difficult to get the analytical expressions except few special cases. The analytical expressions are very important for investigating the dynamical behaviors of physical systems. In this section, we consider two classes of N-qubit states which allow analytical expressions of GQD. We need two mathematical facts below which can be found in many textbooks.

Lemma 3 *(Group homomorphism of $U(2)$ to $SO(3)$). (See [20].) For any two-dimensional unitary matrix U, there exists a unique real*