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Scattering of a relativistic scalar particle by a cusp potential

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Abstract

We solve the Klein–Gordon equation in the presence of a spatially one-dimensional cusp potential. The scattering solutions are obtained in terms of Whittaker functions and the condition for the existence of transmission resonances is derived. We show the dependence of the zero-reflection condition on the shape of the potential. In the low-momentum limit, transmission resonances are associated with half-bound states. We express the condition for transmission resonances in terms of the phase shifts.

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The study of low-momentum scattering of nonrelativistic particles by one-dimensional potentials is a well studied and understood problem [1]. Here we have that, as momentum goes to zero, the reflection coefficient goes to unity unless the potential $V(x)$ supports a zero-energy resonance. In this case the transmission coefficient goes to unity, becoming a transmission resonance [2]. Recently, this result has been generalized to the Dirac equation [3], showing that transmission resonances at $k=0$ in the Dirac equation take place for a potential barrier $V=V(x)$ when the corresponding potential well $V=-V(x)$ supports a supercritical state. Kennedy [4] has shown that this result is also valid for a Woods–Saxon potential. More recently, transmission resonances and half-bound states have been discussed for a Dirac particle scattered by a cusp potential [5,6] as well as for a class of short-range potentials [7]. The bound states for scalar relativistic particles satisfying the Klein–Gordon equation are qualitatively different from the previous case. Here, for short-range attractive potentials the Schiff–Snyder effect [8–14] takes place, i.e. for a given potential strength two bound states appear, one with positive norm and another with negative norm. Such states can be associated with a particle–antiparticle creation process. No antiresonant states appear [11,12].

The absence of resonant overcritical states for the Klein–Gordon equation in the presence of short-range potential interactions does not prevent the existence of transmission resonances for given values of the potential.

Quantum effects associated with scalar particles in the presence of external potentials have been extensively discussed in the literature [10,14]. Among quantum effects, we have that transmission resonance is one of the most interesting phenomenon. For given values of the energy and the proper choice of the shape of the effective barrier, the probability of transmission reaches a maximum such as that obtained in the study of superradiance [14], where the amplitude of the scattered solutions by a rotating Kerr black hole is even larger than the amplitude of the incident wave. Analogous phenomena can also be obtained due to the presence of strong electromagnetic potentials [15].

Recently, transmission resonances for the Klein–Gordon equation in the presence of a Woods–Saxon potential barrier have been computed [16]. The transmission coefficient as a function of the energy and the potential amplitude shows a behavior that resembles the one obtained for the Dirac equation [4]. This result also holds for the square potential [11].

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In this Letter we discuss the scattering of a Klein–Gordon scalar particle by the vector cusp potential [5]

$$eA^0(x) = V(x) = \begin{cases} V_0 e^{x/a} & \text{for } x < 0, \\ V_0 e^{-x/a} & \text{for } x > 0. \end{cases} \quad (1)$$

The potential (1) vanishes exponentially for large values of x, the parameter V_0 determines the strength of the barrier and the constant a defines the width of the potential. The cusp potential (1) reduces to a repulsive delta interaction of strength g in the limit $2aV_0 \to g$ as $a \to 0$. It is the purpose of the present Letter to compute the scattering solutions of the one-dimensional Klein–Gordon equation in the presence of the cusp vector potential and show that one-dimensional scalar wave solutions exhibit transmission resonances with a functional dependence on the shape and strength of the potential similar that obtained for the Dirac equation [4]. The cusp vector potential (1) does not possess a square barrier limit and consequently the phase shift δ, associated with the transmission amplitude, cannot be directly identified with the positions of the transmission resonances [11].

The one-dimensional Klein–Gordon equation, minimally coupled to a vector potential A^μ can be written as

$$\eta^{\alpha\beta}(\partial_\alpha + ieA_\alpha)(\partial_\beta + ieA_\beta)\phi + \phi = 0, \quad (2)$$

where the metric $\eta^{\alpha\beta} = \mathrm{diag}(1, -1)$ and here and thereafter we choose to work in natural units $\hbar = c = m = 1$.

Since the potential $V(x)$ in Eq. (1) does not depend on time, we have that $\phi = \phi(x)\exp(-iEt)$, and the problem of solving the one-dimensional Klein–Gordon equation (2) reduces to that of finding solutions to the second-order differential equation [10]

$$\frac{d^2\phi(x)}{dx^2} + \left[\left(E - V(x)\right)^2 - 1\right]\phi(x) = 0. \quad (3)$$

Let us consider the scattering solutions for $x < 0$ with $E^2 > 1$ of the Klein–Gordon equation. We proceed to solve the differential equation

$$\frac{d^2\phi_L(x)}{dx^2} + \left[\left(E - V_0 e^{x/a}\right)^2 - 1\right]\phi_L(x) = 0. \quad (4)$$

On making the variable change $y = 2iaV_0 e^{x/a}$, Eq. (4) becomes

$$y\frac{d}{dy}\left(y\frac{d\phi_L}{dy}\right) - \left[(iaE - y/2)^2 + a^2\right]\phi_L = 0. \quad (5)$$

Setting $\phi_L = y^{-1/2} f(y)$, Eq. (5) reduces to the Whittaker equation [17, p. 505]

$$\frac{d^2 f(y)}{dy^2} + \left[-\frac{1}{4} + \frac{iaE}{y} + \frac{1/4 - \mu^2}{y^2}\right] f(y) = 0. \quad (6)$$

The general solution of Eq. (6) can be written as

$$\phi_L(y) = c_1 y^{-1/2} M_{\kappa,\mu}(y) + c_2 y^{-1/2} M_{\kappa,-\mu}(y), \quad (7)$$

where $M_{\kappa,\mu}(y)$ is the Whittaker functions [17, p. 505] and

$$\kappa = iaE, \qquad \mu = ia\sqrt{E^2 - 1}. \quad (8)$$

Now we consider the solution for $x > 0$. In this case, the differential equation to solve is

$$\frac{d^2\phi_R(x)}{dx^2} + \left[\left(E - V_0 e^{-x/a}\right)^2 - 1\right]\phi_R(x) = 0. \quad (9)$$

On making the variable change $z = 2iaV_0 e^{-x/a}$, Eq. (9) can be written as

$$z\frac{d}{dz}\left(z\frac{\phi_R}{dz}\right) - \left[(iaE - z/2)^2 + a^2\right]\phi_R = 0. \quad (10)$$

Putting $\phi_R = z^{-1/2} g(z)$ we obtain the Whittaker differential equation

$$\frac{d^2 g(z)}{dz^2} + \left[-\frac{1}{4} + \frac{iaE}{z} + \frac{1/4 - \mu^2}{z^2}\right] g(z) = 0 \quad (11)$$

whose solution is

$$\phi_R(z) = c_3 z^{-1/2} M_{\kappa,-\mu}(z) + c_4 z^{-1/2} M_{\kappa,\mu}(z). \quad (12)$$

Using the asymptotic behavior of the Whittaker function $M_{\kappa,\mu}(y) \to e^{-y/2} y^{1/2+\mu}$, as $y \to 0$ [17, p. 504], we can write the incoming wave solution $\phi_{\mathrm{inc}}(x)$ in the form

$$\phi_{\mathrm{inc}}(x) = c_1 (2iaV_0)^{-1/2} e^{-x/2a} M_{\kappa,\mu}\left(2iaV_0 e^{x/a}\right). \quad (13)$$

As $x \to -\infty$, ϕ_{inc} behaves like a plane wave traveling to the right

$$\phi_{\mathrm{inc}} \to c_1 (2iaV_0)^{\mu} e^{i\sqrt{E^2-1}x}. \quad (14)$$

Analogously, we have that the reflected $\phi_{\mathrm{ref}}(x)$ solution can be written as

$$\phi_{\mathrm{ref}}(x) = c_2 (2iaV_0)^{-1/2} e^{-x/2a} M_{\kappa,-\mu}\left(2iaV_0 e^{x/a}\right). \quad (15)$$

As $x \to -\infty$, $\phi_{\mathrm{ref}}(x)$ has the asymptotic behavior

$$\phi_{\mathrm{ref}} \to c_2 (2iaV_0)^{-\mu} e^{-i\sqrt{E^2-1}x}. \quad (16)$$

Finally, using the right solution ϕ_R (12), we have that the transmitted solution $\phi_{\mathrm{trans}}(x)$ can be expressed as

$$\phi_{\mathrm{trans}}(x) = c_3 (2iaV_0)^{-1/2} e^{x/2a} M_{\kappa,-\mu}\left(2iaV_0 e^{-x/a}\right), \quad (17)$$

with $c_4 = 0$. As $x \to \infty$, $\phi_{\mathrm{trans}}(x)$ takes the asymptotic plane wave behavior

$$\phi_{\mathrm{trans}} \to c_3 (2iaV_0)^{-\mu} e^{i\sqrt{E^2-1}x}. \quad (18)$$

The electrical current density for the one-dimensional Klein–Gordon equation (2) is given by the expression:

$$j^\mu = \frac{i}{2}\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right). \quad (19)$$

The current as $x \to -\infty$ can be decomposed as $j_L = j_{\mathrm{in}} - j_{\mathrm{refl}}$ where j_{in} is the incident current and j_{refl} is the reflected one. Analogously we have that, on the right side, as $x \to \infty$ the current is $j_R = j_{\mathrm{trans}}$, where j_{trans} is the transmitted current.

Using the reflected j_{refl} and transmitted j_{trans} currents, we have that the reflection and transmission coefficients R and T can be expressed as

$$R = \frac{j_{\mathrm{refl}}}{j_{\mathrm{inc}}}, \qquad T = \frac{j_{\mathrm{trans}}}{j_{\mathrm{inc}}}. \quad (20)$$