Physics Letters A 373 (2009) 735–740

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Harmonic balancing approach to nonlinear oscillations of a punctual charge in the electric field of charged ring

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ARTICLE INFO

Article history:
Received 10 November 2008
Received in revised form 5 December 2008
Accepted 18 December 2008
Available online 31 December 2008
Communicated by A.R. Bishop

Keywords:
Nonlinear oscillator
Harmonic balance method
Approximate frequency

ABSTRACT

The harmonic balance method is used to construct approximate frequency–amplitude relations and periodic solutions to an oscillating charge in the electric field of a ring. By combining linearization of the governing equation with the harmonic balance method, we construct analytical approximations to the oscillation frequencies and periodic solutions for the oscillator. To solve the nonlinear differential equation, firstly we make a change of variable and secondly the differential equation is rewritten in a form that does not contain the square-root expression. The approximate frequencies obtained are valid for the complete range of oscillation amplitudes and excellent agreement of the approximate frequencies and periodic solutions with the exact ones are demonstrated and discussed.

1. Introduction

The purpose of this short communication is to determine the high order periodic solutions to the oscillations of a charge in the electric field of a charged ring by applying the harmonic balance method [1–9]. To do this, we use the analytical approach developed by Lim and Wu [2,3] in which linearization of the governing equation is combined with the harmonic balance method. This is an example of a strongly nonlinear oscillator for which the linearization of the governing equation is only valid for very small values of the displacement. To our best knowledge, this is the first time the harmonic balance method is applied to this type of nonlinear oscillator.

2. Solution procedure

We consider a ring of radius R with a charge $Q > 0$ spread uniformly around the ring. The electric field E on the axis (say the x-axis) of the ring is given by

$$E(x) = \frac{1}{4\pi\varepsilon_0}\frac{Qx}{(R^2+x^2)^{3/2}} \tag{1}$$

where x is the distance along the axis. If a negative punctual charge $q = -|q|$ is placed at a point on the ring axis, the charge will experience a force

$$F(x) = -\frac{1}{4\pi\varepsilon_0}\frac{|q|Qx}{(R^2+x^2)^{3/2}}. \tag{2}$$

The equation of motion of the punctual charge q is given by the following nonlinear differential equation

$$m\frac{d^2x}{dt^2} + \frac{1}{4\pi\varepsilon_0}\frac{|q|Qx}{(R^2+x^2)^{3/2}} = 0 \tag{3}$$

with initial conditions

$$x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(0) = 0. \tag{4}$$

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.12.042

Even though an oscillating electric charge, because of its acceleration, continuously radiates and dissipates energy [10], we suppose that the fractional change of the total energy per oscillation due to radiation is negligibly small.

Eq. (3) can be written as follows

$$\frac{1}{R}\frac{d^2x}{dt^2}+\omega_0^2\left(1+\frac{x^2}{R^2}\right)^{-3/2}\frac{x}{R}=0 \tag{5}$$

where

$$\omega_0=\sqrt{\frac{|q|Q}{4\pi\varepsilon_0 mR^3}}. \tag{6}$$

Two dimensionless variables y and τ can be constructed as follows

$$x=Ry \quad \text{and} \quad t=\omega_0\tau. \tag{7}$$

Substituting these dimensionless variables into Eq. (4) gives

$$\frac{d^2y}{d\tau^2}+\frac{y}{(1+y^2)^{3/2}}=0 \tag{8}$$

with initial conditions

$$y(0)=A \quad \text{and} \quad \frac{dy}{d\tau}(0)=0 \tag{9}$$

where $A=x_0/R$, being x_0 the initial position of the punctual charge q.

Eq. (8) is an example of a conservative highly nonlinear oscillatory system in which the restoring force has an irrational form. All the motions corresponding to Eq. (8) are periodic; the system will oscillate within symmetric bounds $[-A, A]$, and the angular frequency ω and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude A.

For small x, Eq. (8) approximates that of a linear harmonic oscillator

$$\frac{d^2y}{d\tau^2}+y=0 \tag{10}$$

so, for large A, we have $\omega\approx 1$. For large x, Eq. (8) approximates that of a truly nonlinear oscillator

$$\frac{d^2y}{d\tau^2}+\frac{\mathrm{sgn}(y)}{y^2}=0 \tag{11}$$

and ω tends to zero when A increases.

It is difficult to solve nonlinear differential equations and, in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear oscillatory system [1,11]. There are many approaches for approximating solutions to nonlinear oscillatory systems. The most widely studied approximation methods are the perturbation methods [1]. The simplest and perhaps one of the most useful of these approximation methods is the Lindstedt–Poincaré perturbation method, whereby the solution is analytically expanded in the power series of a small parameter. To overcome this limitation, many new perturbative techniques have been developed. Modified Lindstedt–Poincaré techniques, homotopy perturbation method or linear delta expansion are only some examples of them. A recent detailed review of perturbation methods can be found in Refs. [11] and [12].

The harmonic balance method is another procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation [1]. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required. Since the restoring force is an odd function of y, the periodic solution $y(\tau)$ has the following Fourier series representation [1]

$$y(\tau)=\sum_{n=0}^{\infty}h_{2n+1}(A)\cos\big[(2n+1)\omega_e(A)\tau\big] \tag{12}$$

which contains only odd multiples of $\omega_e\tau$. In Eq. (12), ω_e is the exact frequency of the nonlinear oscillator. The purpose of the harmonic balance method is to approximate the periodic solution in Eq. (12) by a trigonometric polynomial [1]

$$y(\tau)\approx\sum_{n=0}^{N}b_{2n+1}(A)\cos\big[(2n+1)\omega(A)\tau\big] \tag{13}$$

and determine both the coefficients b_{2n+1} and the approximate angular frequency ω as a function of A. The different approximation orders are obtained by choosing $N=0$ (first-order—one harmonic, $\cos\omega\tau$), $N=1$ (second-order—two harmonics, $\cos\omega\tau$ and $\cos 3\omega\tau$), ..., in Eq. (13).

The main objective of this research is to solve Eq. (8) by applying the harmonic balance method, and to compare the approximate frequency obtained with the exact one. The approximate frequency derived here is accurate and closer to the exact solution. To do this, Eq. (8) is rewritten in a form that does not contain the square-root expression

$$\left(1+y^2\right)^3\left(\frac{d^2y}{d\tau^2}\right)^2-y^2=0. \tag{14}$$