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Non-isospectral integrable couplings of Ablowitz–Ladik hierarchy with self-consistent sources

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ABSTRACT

A kind of new non-isospectral integrable couplings of discrete soliton equations hierarchy with self-consistent sources associated with $\widetilde{sl}(4)$ is presented. As an application example, the integrable coupling hierarchy of non-isospectral Ablowitz–Ladik with self-consistent sources is derived by using of the loop algebra $\widetilde{sl}(4)$.

1. Introduction

Soliton equations with self-consistent sources [1–6] may have received considerable attention in recent years. Physically, the sources may result in solitary waves moving with a non-constant velocity and therefore lead to a variety of dynamics of physical models. For applications, these kind of systems are usually used to describe interactions between different solitary waves and are relevant to some problems of hydrodynamics, solid state physics, plasma physics, etc.

Recently, the integrable couplings have been receiving growing attention. A few ways to establish integrable couplings are presented by using perturbations [7–9], enlarging spectral problems [10,11], semi-direct sums of Lie algebras [12,13], and creating new Lie algebras [14–20]. Moreover, the Hamiltonian forms of the integrable nonlinear equations can be constructed by the bi-trace identity [21], the continuous variational identity was presented by using of the semi-direct sums of Lie algebras in [22], and the bi-Hamiltonian formulation for triangular systems by perturbations was constructed in [23], which are beautiful results in soliton theory.

Non-isospectral evolution equations may also be of physical and mathematical importance. They may be related to time-dependent spectral parameters and may describe solitary waves in non-uniform media [24,25]. Meanwhile, the time-dependent spectral parameters will lead to generalizations [26,27] of those classical methods. The algebraic structure of discrete zero curvature equations and master symmetries of discrete evolution equations were presented in [28], which is an important work to help discussing non-isospectral lattice equations.

In particular, AKNS, MKdV and KdV equation hierarchies with self-consistent sources have been presented in [29]. Complexiton solutions of the Korteweg–de Vries equation self-consistent sources, and soliton and positon solutions of the Schrödinger source equation were presented [5,6]. However, these methods are not designed for constructing the non-isospectral integrable couplings of discrete equations hierarchy with self-consistent sources. Hence, we consider to establish a kind of the non-isospectral integrable coupling system with self-consistent sources by using of loop algebra $\widetilde{sl}(4)$.

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2. Non-isospectral zero curvature equation

Consider the following non-isospectral linear problem

$$\begin{cases} E\psi_n = U_n\psi_n, \\ \psi_{nt} = \omega(\lambda)\psi_{ny} + V_n\psi_n, \end{cases} \tag{1}$$

where $U_n = e_0 + u_1e_1 + \cdots + u_pe_p$, $u_i(n,t,y) = u_i(n)$ $(i = 1,2,\ldots,p)$, $\psi(n) = \psi(n,t,y)$ are field variables defining on $n \in Z$, $t \in R$; $e_i = e_i(\lambda) \in sl(4)$; $sl(4)$ denotes a finite-dimensional Lie algebra over C; The operator E is defined by $Ef_n = f_{n+1}$. Then, the spectral parameter $\lambda = \lambda(t,y)$ satisfies a non-isospectral condition of the form

$$\lambda_t = \omega(\lambda)\lambda_y + \beta(\lambda), \tag{2}$$

where $\omega(\lambda)$ and $\beta(\lambda)$ are two functions to be specified. Now let us discuss the spectral problem (1). The compatibility condition of (1) gives

$$\frac{\partial U_n}{\partial t} + \frac{\partial U_n}{\partial \lambda}\frac{\partial \lambda}{\partial t} - \omega(\lambda)\left[\frac{\partial U_n}{\partial y} + \frac{\partial U_n}{\partial \lambda}\frac{\partial \lambda}{\partial y}\right] = V_{n+1}U_n - U_nV_n. \tag{3}$$

It is a new $(2+1)$-dimensional non-isospectral discrete zero curvature equation.

When $\frac{\partial\lambda}{\partial t} = 0$, $\frac{\partial\lambda}{\partial y} = 0$ and $\frac{\partial U_n}{\partial y} = 0$, we obtain the well-known discrete zero curvature equation

$$U_{nt} = (EV_n)U_n - U_nV_n. \tag{4}$$

When $\frac{\partial\lambda}{\partial t} \neq 0$, $\frac{\partial\lambda}{\partial y} = 0$ and $\frac{\partial U_n}{\partial y} = 0$, we obtain the non-isospectral discrete zero curvature equation

$$U_{nt} + \frac{\partial U_n}{\partial \lambda}\frac{\partial \lambda}{\partial t} = (EV_n)U_n - U_nV_n. \tag{5}$$

3. A kind of non-isospectral integrable couplings of discrete soliton equation hierarchy with self-consistent sources

In the following, we consider a set of Lie algebra $sl(4)$ [22]

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{6a}$$

$$e_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6b}$$

It is easy to see that

$$sl(4) = \mathrm{span}\{e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8\}, \qquad sl(4)_1 = \mathrm{span}\{e_1,e_2,e_3,e_4\}, \qquad sl(4)_2 = \mathrm{span}\{e_5,e_6,e_7,e_8\}$$

construct three Lie algebra, and $sl(4) = sl(4)_1 \oplus sl(4)_2$, $[sl(4)_1, sl(4)_2] = sl(4)_1sl(4)_2 - sl(4)_2sl(4)_1 \subseteq sl(4)_2$. In what follow, we introduce the loop algebra $\widetilde{sl}(4)$

$$\widetilde{sl}(4) = \{A \mid A \in R[\lambda] \otimes sl(4)\}, \qquad \widetilde{sl}_1(4) = \{A \mid A \in R[\lambda] \otimes sl(4)_1\}, \qquad \widetilde{sl}(4)_2 = \{A \mid A \in R[\lambda] \otimes sl(4)_2\}, \tag{7}$$

where the loop algebra $\widetilde{sl}(4)$ is define by span $\{\lambda^n A \mid n \geqslant 0, A \in sl(4)\}$. As a semi-direct sum of loop algebra $\widetilde{sl}(4)_1$ and $\widetilde{sl}(4)_2$, the loop algebra $\widetilde{sl}(4)_2$ is an Abelian ideal of the loop algebra $\widetilde{sl}(4)$.

Consider the auxiliary linear problem

$$E\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = U(u,\lambda)\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}, \qquad U(u,\lambda) = \left[e_1(\lambda) - e_2(\lambda)\right] + \sum_{i=1}^{8} u_ie_i(\lambda), \qquad \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}_{t_n} = V_n(u,\lambda)\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}, \tag{8}$$

where $u = (u_1,\ldots,u_s)^T$, $U_n = e_0 + u_1e_1 + \cdots + u_pe_p$, $u_i(n,t) = u_i$ $(i = 1,2,\ldots,p)$, $\phi_i = \phi(n,t)$ are field variables defining on $n \in R$, $t \in R$; $e_i = e_i(\lambda) \in sl(4)$; $sl(4)$ denotes a finite-dimensional Lie algebra over C. Then, the spectral parameter $\lambda = \lambda(t,n)$ satisfies a non-isospectral condition of the form

$$\lambda_t = \beta(\lambda), \tag{9}$$

where $\beta(\lambda)$ is a function to be specified. From the spectral problem (8), the compatibility condition of (8) gives

$$\frac{\partial U_n}{\partial t} + \frac{\partial U_n}{\partial \lambda}\frac{\partial \lambda}{\partial t} - (EV_n)U_n + U_nV_n = 0, \tag{10}$$

which is a non-isospectral zero curvature equation.

When $\frac{\partial\lambda}{\partial t} = 0$, Eq. (10) gives rise to the well-known zero curvature equation

$$U_{nt} - (EV_n)U_n + U_nV_n = 0, \quad n = 1,2,\ldots. \tag{11}$$