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New embedded pairs of explicit Runge–Kutta methods with FSAL properties adapted to the numerical integration of oscillatory problems

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ABSTRACT

A higher order Runge–Kutta (pair) method specially adapted to the numerical integration of IVPs with oscillatory solutions is presented. This method is based on the adapted methods proposed by Franco (see Ref. [J.M. Franco, Appl. Numer. Math. 50 (2004) 427]). We give explicit method (up to order 5) as well as pairs of embedded Runge–Kutta methods of order 5 and 4 designed using the FSAL properties. The stability of the new methods is analyzed. The numerical experiments are carried out to show the efficiency and robustness of our methods in comparison with some efficient methods.

1. Introduction

In this Letter we are concerned with the numerical integration of oscillatory problems modeled by initial value problems of the form

$$y' = f(x, y), \qquad y(x_0) = y_0, \quad x \in [x_0, X], \tag{1}$$

where whose solution exhibits a pronounced oscillatory character. This kinds of problems often arise in many fields of applied science such as mechanics, astrophysics, quantum chemistry, electronics. They can be integrated with general purpose methods or using other codes adapted to the special structure of the problem. In [1], Bettis derived explicit Runge–Kutta-type algorithm with 3 and 4 stages for the integration of ODEs with oscillatory solutions. Basing on the rooted trees and B-series theory, furthermore, Franco gave the necessary and sufficient order conditions for this class of Runge–Kutta methods, and constructed explicit methods (up to order 4) as well as the pairs of embedded Runge–Kutta methods of orders 4 and 3 (see Ref. [2]). This motivates some new and robust methods with higher order.

In this Letter, therefore, we present a new Runge–Kutta method of order 5 as well as pairs of embedded Runge–Kutta methods of orders 5 and 4 adapted to the numerical integration of oscillatory problems designed using the FSAL technique (the last evaluation at any step is the same as the first evaluation at the next step). The numerical experiments are reported.

2. Preliminaries and some basic results

In [2], Franco constructed a formula of the Adapted RK methods for solving oscillatory ODEs. This method can be introduced simply as follows.

Applying an s-stage explicit RK method (A, b, c) to the test differential equation

$$y' = i\omega y, \quad \omega \in \mathbb{R}, \tag{2}$$

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gives

$$y_{n+1} = R(i\omega h) y_n, \tag{3}$$

where

$$R(z) = 1 + zb^T(I - zA)^{-1}e = 1 + zb^T\big(I + zA + z^2A^2 + \cdots\big)e = 1 + zb^Te + z^2b^TAe + z^3b^TA^2e + \cdots + z^sb^TA^{s-1}e. \tag{4}$$

Since $y(x) = e^{i\omega x}y_0$ is the exact solution of (2), we have

$$y_{n+1} = e^{i\omega h} y_n. \tag{5}$$

If the explicit RK method is of order p, then $R(z) = e^z + \mathcal{O}(z^{p+1})$, thus we have

$$b^TA^je = \frac{1}{(j+1)!}, \quad j = 0, 1, \ldots, p-1. \tag{6}$$

Theorem 1. *(See Ref. [2].) The necessary and sufficient conditions for an adapted RK method to be of order p are given by*

$$b^T(v)\Phi(\tau) - \frac{1}{\gamma(\tau)} = \mathcal{O}\big(v^{p+1-\rho(\tau)}\big) = \mathcal{O}\big(h^{p+1-\rho(\tau)}\big), \quad \rho(\tau) = 1, 2, \ldots, p, \tag{7}$$

where $v = h\omega$, τ expresses a rooted tree and the functions $\rho(\tau)$, $\alpha(\tau)$, $\gamma(\tau)$ and $\Phi(\tau)$ are defined in Refs. [3,4].

Theorem 2. *(See Ref. [2].) If we replace the order conditions that correspond to the high trees*

$$b(v)^TA^je - \frac{1}{(j+1)!} = \mathcal{O}\big(v^{p-j}\big), \quad j = 0, 1, \ldots, p-1, \tag{8}$$

with the following conditions

$$b^TA^je = \frac{1}{(j+1)!}, \quad j = 0, 1, \ldots, p-3, \qquad b^TA^{p-2}e = \phi_{p-1,s}(v), \qquad b^TA^{p-1}e = \phi_{p,s}(v), \tag{9}$$

where $\phi_{p-1,s}(v)$ and $\phi_{p,s}(v)$ are defined by

$$\phi_{j,s}(v) = \begin{cases} \phi_j(v), & \text{if } s = j \text{ or } j+1, \\ \phi_j(v) + v^2(b^TA^{j+1}e) + \cdots + (-1)^\beta v^{2\beta+2}(b^TA^{j+2\beta+1}), & \text{if } s \geqslant j+2, \end{cases} \tag{10}$$

with $\beta = [\frac{s-j-2}{2}]$, and the ϕ-functions are defined in Ref. [1], then the resulting Adapted RK method is of order p.

The ϕ-functions defined in [1] have the following recurrence relation:

$$\phi_0(v) = \cos(v), \qquad \phi_1(v) = \sin(v)/v, \qquad \phi_{j+2}(v) = \left(\frac{1}{j!} - \phi_j(v)\right)\Big/v^2, \quad j \geqslant 0.$$

In order to avoid the possible round-off error (when $|v|$ is small), the ϕ-functions can also be replaced by the Taylor expansions

$$\phi_j(v) = \sum_{k=0}^{\infty}(-1)^k\frac{v^{2k}}{(2k+j)!}, \quad j \geqslant 0.$$

3. Construction of a higher order explicit adapted RK methods using the FSAL technique

In [2], Franco constructed a formula of the Adapted RK methods for solving ODEs with oscillating solutions. In a slightly different way, we will construct a higher order adapted RK method (up to order 5) as well as pairs of embedded Runge–Kutta methods of orders 5 and 4 based on the Dormand–Prince 5(4) in [5] designed using the FSAL technique.

3.1. A fifth-order adapted RK method

We consider the six-stages explicit RK method displayed by the following Butcher-tableau:

0	0					
1/5	1/5	0				
3/10	3/40	9/40	0			
4/5	44/45	−56/15	32/9	0		
8/9	19 372/6561	−25 360/2187	64 448/6561	−212/729	0	
1	9017/3168	−355/33	46 732/5247	49/176	−5103/18 656	0
	$b_1(v)$	$b_2(v)$	$b_3(v)$	$b_4(v)$	$b_5(v)$	$b_6(v)$

The entries of A and C are chosen from the well known five-order RK method derived by Dormand–Prince in Ref. [5]. Because the entries of A and C are available, we need only to find the weights of the adapted RK method from the order conditions that correspond to homogeneous linear problems (see [2]). With stage $s = 6$ and order $p = 5$ it follows that

$$b^Te = 1, \qquad b^TAe = 1/2, \qquad b^TA^2e = 1/6, \qquad b^TA^3e = \phi_4(v) + v^2\big(b^TA^5e\big), \qquad b^TA^4e = \phi_5(v),$$

in which $v = \omega h$, and the ϕ-functions are defined in [11].