Physics Letters A 372 (2008) 6593–6601

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Statistical independence in nonlinear maps coupled to non-invertible transformations

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ARTICLE INFO

Article history:
Received 19 March 2008
Received in revised form 30 July 2008
Accepted 28 August 2008
Available online 2 September 2008
Communicated by A.R. Bishop

PACS:
05.45.+b

Keywords:
Asymptotic deterministic randomness
Statistical independence
Lisssajous maps

ABSTRACT

We investigate the connections between functions of type $x_n = p(\theta T z^n)$ and nonlinear maps coupled to non-invertible transformations. These systems can produce unpredictable dynamics. We study the higher-order correlations in the generated sequences. We show that (theoretically) it is possible to construct systems that can generate sequences that constitute a set of statistically independent random variables. We apply the results in the improvement of a two-dimensional coupled map system that has been used in practical applications as e.g. cryptosystems and data compression.

1. Introduction

In science, chaos theory describes the behavior of nonlinear systems that, for some values of the parameters, exhibit dynamics that are sensitive to initial conditions. We should stress that the normal chaotic systems are deterministic in the sense that, in the dynamics of a given variable, future values are determined by previous values [1–12]. The normal chaotic systems produce irregular dynamics, but they are not completely random. In fact, using past values, it is possible to make predictions in the short term. In truly random systems, even the next value is unpredictable [12–32].

One of the paradigmatic models of chaos theory is the logistic map [3–9]: $x_{n+1} = ux_n(1 - x_n)$. Recently the theory of functions and dynamical systems that can produce unpredictable time-series has been a very active area of research [15–28]. Some of these works are based on ideas as the following. Let us define the function:

$$x_n = p\left(\theta T z^n\right) \tag{1}$$

where $p(\bullet)$ is a periodic function, θ and z are real numbers and T is the period of function $p(\bullet)$. For an integer $z > 1$, Eq. (1) are the solutions to several classes of chaotic maps [15–21,29–35].

On the other hand, Eq. (1) for most non-integer z cannot be expressed as a map of type $x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-r+1})$. Given any string of past values $x_n, x_{n-1}, \dots, x_{n-r+1}$, of any length, the next value x_{n+1} cannot be predicted [15–20].

Several dynamical systems have been proposed to generate similar dynamics [17–28]. In the present Letter we will investigate the higher-order correlations of nonlinear maps coupled to non-invertible transformations. We will propose a dynamical system with very good statistical properties which can be applied in the simulation of random processes.

2. Properties of functions $x_n = p(\theta T z^n)$

Recently several papers have been dedicated to the investigation of functions given by Eq. (1). The exact general solution to many other maps can be obtained using functions of type $x_n = p[\theta T k^n]$, where k is an integer number [15–20,31–35]. For a non-integer z, the dynamics generated by Eq. (1) can be unpredictable from the previous values [15–20]. We will illustrate some of the properties of these functions using the example

$$x_n = \cos\left[2\pi\theta z^n\right]. \tag{2}$$

Let z be a rational number expressed as $z = p/q$, where p and q are relative prime numbers. If we have $m + 1$ numbers generated by Eq. (2): $x_0, x_1, \dots, x_m$ (m can be as large as we wish), the next value x_{m+1} is still unpredictable. This is correct for any string of $m + 1$ values. In general, x_{m+1} can take q different values. Fig. 1

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.08.054

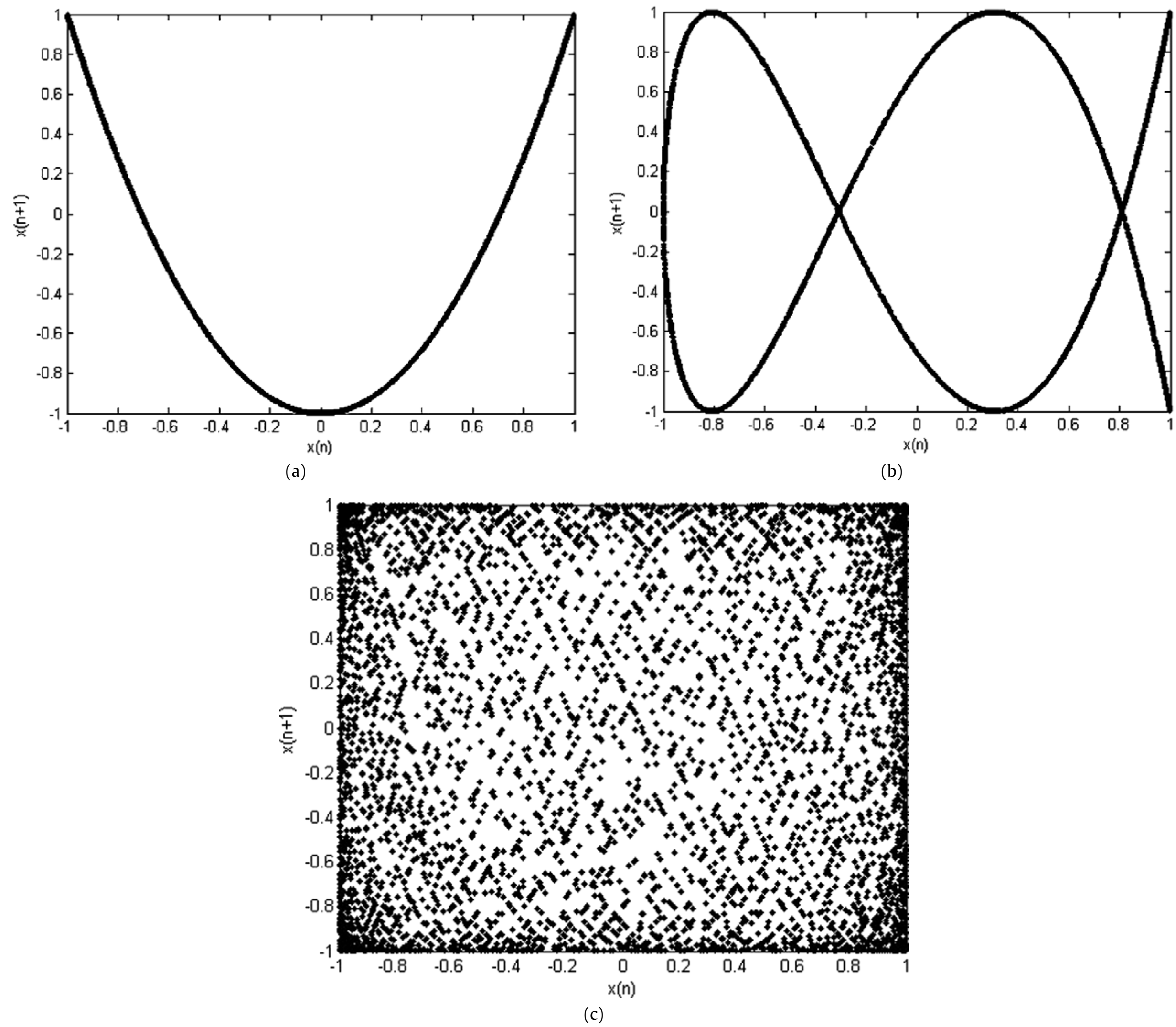

Fig. 1. First return maps generated by Eq. (2): (a) $z = 2$, (b) $z = 5/2$, (c) $z = e$.

shows the first return maps constructed using the time series generated by Eq. (2) with different values of parameter z.

The probability densities of these process will depend on the specific function $p(\bullet)$ used in Eq. (1) [19]. For instance, for the function given by Eq. (2), the probability is $\eta(x) = 1/(\pi\sqrt{1-x^2})$. On the other hand, for the function $x_n = \theta z^n \mod 1$, the probability density is $\eta(x) = 1$. Fig. 2 shows the first-return map produced by the dynamics of this function.

3. Dynamical systems: Lissajous maps

The dynamics produced by Eq. (1) presents fundamental differences with the previously known chaotic systems. This phenomenon is called "deterministic randomness" [18,24–28]. A very interesting development in the study of "deterministic randomness" has been the construction of the so-called Lissajous maps [24–28]. The Lissajous maps are defined as follows: $x_{n+1} = f(zf^{-1}(x_n))$, $y_{n+1} = f(bf^{-1}(x_{n+1}))$, where $f(\bullet)$ is a periodic function, z and b are real parameters. Given some conditions for z and b, the Lissajous maps have been shown to produce exactly the return maps that are generated by the random processes defined by Eq. (1) [24–28]. Consider the following particular example: $x_{n+1} = \cos[z \arccos x_n]$, $y_{n+1} = \cos[b \arccos x_{n+1}]$. Suppose $z = p/q$, where p and q are relative prime numbers and $b = q^N$. When $m = 0, 1, \ldots, N$, the mth return map represents a perfect Lissajous figure. The sequence generated by this system is unpredictable in N steps [28]. When $b \to \infty$, this system is equivalent to the process $y_n = \cos[2\pi z^n]$ [28]. Fig. 3 shows different realizations of the dynamics generated by this system.

4. Higher-order correlations

In this section we will follow the ideas of papers [19,36–40] for the investigation of the *r-order* correlations of sequences. The *r-order* correlations are defined as follows:

$$E(x_{n_1}x_{n_2}\cdots x_{n_r}) = \int_{-1}^{1}\left[\eta(x_0)x_{n_1}x_{n_2}\cdots x_{n_r}\right]dx_0 \tag{3}$$

where $\eta(x_0) = 1/(\pi\sqrt{1-x_0^2})$.

Functions $f_1 f_2 \cdots f_r$ constitute a set of statistically independent functions if and only if

$$E\left(f_1^{k_1} f_2^{k_2}\cdots f_r^{k_r}\right) = E\left(f_1^{k_1}\right)E\left(f_2^{k_2}\right)\cdots E\left(f_r^{k_r}\right) \tag{4}$$

for all positive integers $k_1, k_2, \ldots, k_r$. For a transcendental z, we will show that the sequences $x_{n_0}x_{n_1}\cdots x_{n_r}$ constitute a set of statistically independent random variables if all $n_0, n_1, \ldots, n_r$ are different. The following condition is satisfied,

$$E\left(x_{n_0}^{k_0}x_{n_1}^{k_1}\cdots x_{n_r}^{k_r}\right) = E\left(x_{n_0}^{k_0}\right)E\left(x_{n_1}^{k_1}\right)\cdots E\left(x_{n_r}^{k_r}\right) \tag{5}$$