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Periods of relativistic oscillators with even polynomial potentials

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ABSTRACT

The authors modify a non-perturbative variational approach based on the Principle of Minimal Sensitivity to calculate the periods of relativistic oscillators with even polynomial potentials. The optimization of the variational parameter is adapted by introducing additional free parameters whose values are set using the ultrarelativistic limit of the period as a boundary condition. Compact general approximations for the potentials $\frac{x^2}{2}+\frac{x^{2m}}{2m}$, $\sum_{n=1}^{m}\frac{x^{2n}}{2n}$ and $\frac{x^{2m}}{2m}$ prove to be accurate over the whole solution domain and even for large values of m.

1. Introduction

The general significance of the study of strongly non-linear systems [1,2] has renewed interest in the relativistic harmonic oscillator [3], whose motion is given non-dimensionally as

$$\ddot{x}+x\left(1-\dot{x}^2\right)^{3/2}=0. \tag{1}$$

The anharmonicity of Eq. (1) corresponds to a transition between physical regimes described by the particle's maximum velocity $\dot{x}_{\max}$. Low energy oscillations at $|\dot{x}_{\max}|\ll 1$ reduce Eq. (1) to simple harmonic motion, while oscillations energetic enough to reach relativistic speeds describe an increasingly non-linear system as $|\dot{x}_{\max}|\to 1$.

Recently, the non-perturbative methods of harmonic balance [4–6] and homotopy perturbation [7,8] have been successfully applied to give analytical approximations of the oscillation period and periodic solution of Eq. (1). Unlike the traditional perturbation theory, these asymptotic techniques do not rely on any linear term or physically small parameter. They are therefore suited to the problem since solutions must be valid for both low and high energy oscillations. The application of these methods yields approximations that have minimal error over the entire domain $|\dot{x}_{\max}|\in[0,1)$. However, their treatment of the relativistic oscillator has thus far been limited to the harmonic potential.

In this work we present a novel method for computing analytical approximations of the period of strongly anharmonic oscillations described by

$$\ddot{x}+\frac{d\phi}{dx}\left(1-\dot{x}^2\right)^{3/2}=0, \tag{2}$$

where $\phi(x)$ is an even polynomial potential. We modify a non-perturbative variational approach that has been developed and applied by Amore et al. in computing the periods of non-relativistic non-linear oscillators [9,10] and the deflection angle in gravitational lensing [11]. This method operates by converting the relevant integral into a series carrying a variational parameter ω, which can be chosen using the Principle of Minimal Sensitivity (PMS) [12]. In our case the standard procedure for choosing ω is adapted by treating the existing harmonics as free parameters. These are then assigned values by imposing the ultrarelativistic limit of the period as a boundary condition. The modification is able to treat general forms of even polynomial potentials, broadly extending the class of treatable non-linear systems. This includes potentials that are non-linear even in the low energy regime and also those that do not carry a linear term at all. The resulting ap-

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proximations are compact and accurate over the whole solution domain.

2. Method

As applied to the relativistic harmonic oscillator [4], the qualitative analysis of ordinary differential equations detailed in Ref. [13] can be used to show that the solution to Eq. (2) is indeed periodic. That is, the phase space trajectories given by

$$\frac{dy}{dx}+\frac{d\phi}{dx}\frac{(1-y^2)^{3/2}}{y}=0, \tag{3}$$

where $y=\dot{x}$, are closed and have the strip structure:

$$-\infty < x < +\infty \quad \text{and} \quad -1 < y < +1. \tag{4}$$

As with the aforementioned conditions for $\dot{x}_{\max}$, the second inequality above reflects the physical requirement that the speed of the particle is always less than the speed of light.

The period of this system can be derived from the generalized energy–momentum relation

$$E=\sqrt{p^2+1}+\phi(x), \tag{5}$$

where E is the total energy and p is the relativistic momentum. In all of the above equations the rest mass and speed of light are set to unity.

By noting that the total energy is the sum of the potential energy at the oscillation amplitude A and the rest mass, i.e. $E=\phi(A)+1$, it can be shown from Eq. (5) that

$$\phi(A)-\phi(x)+1=\left(1-y^2\right)^{-1/2}. \tag{6}$$

This leads to the expression for the period

$$T=2\int_{-A}^{A} dx\frac{\phi(A)-\phi(x)+1}{\sqrt{[\phi(A)-\phi(x)][\phi(A)-\phi(x)+2]}}. \tag{7}$$

Since the period is a function of A, it would be useful to express the physical regimes described by $\dot{x}_{\max}$ in terms of this parameter. Setting $x=0$ in Eq. (6) gives

$$|\dot{x}_{\max}|=\frac{\sqrt{\phi(A)[2+\phi(A)]}}{1+\phi(A)}. \tag{8}$$

This relation maps the domain $|\dot{x}_{\max}|\in[0,1)$ unto $A\in[0,\infty)$.

For an even potential $\phi(x)$, the period from Eq. (7) can be written as

$$T=2\int_{-A}^{A}\frac{dx}{\sqrt{R(x)(A^2-x^2)}}. \tag{9}$$

We approximate $R(x)(A^2-x^2)$ with the solvable harmonic potential $\omega^2(A^2-x^2)$, in a manner that is essentially a Linear Delta Expansion [11,14]:

$$T=2\int_{-A}^{A}\frac{dx}{\sqrt{[A^2-x^2][\omega^2+\delta(R(x)-\omega^2)]}}. \tag{10}$$

The effective potential in the above equation is linearly interpolated between the harmonic potential (when $\delta=0$) and the original potential of Eq. (9) (when $\delta=1$). In the following calculation Eq. (10) will be evaluated as a power series in δ up to a desired order δ^n, thereafter setting δ back to 1.

With a change of variable $x=A\cos\theta$, Eq. (10) becomes

$$T=\frac{2}{\omega}\int_{0}^{\pi}\frac{d\theta}{\sqrt{1+\delta\Delta(\theta,\omega)}}, \tag{11}$$

where the substitution $\Delta(\theta,\omega)=-1+\frac{R(\theta)}{\omega^2}$ has been made. A binomial expansion of the integrand in powers of δ then gives

$$T=\frac{2}{\omega}\sum_{n=0}^{\infty}\binom{n}{-1/2}\delta^n\int_{0}^{\pi}\Delta^n(\theta,\omega)\,d\theta. \tag{12}$$

Evaluating the above series to a finite order and setting $\delta=1$ will leave an explicit ω-dependence that should be eliminated. Indeed this is the basis of the PMS, requiring that $\frac{\partial T}{\partial\omega}=0$. As shown in Ref. [10], the condition is equivalent to choosing ω so that

$$\int_{0}^{\pi}\Delta^n(\theta,\omega)\,d\theta=0 \tag{13}$$

for an nth order optimization. To first order, the above criterion is equivalently

$$\omega=\sqrt{\langle R(\theta)\rangle}, \tag{14}$$

where $\langle R(\theta)\rangle$ denotes the mean value of $R(\theta)$.

For non-relativistic oscillators Eq. (14) is implemented easily, yielding highly accurate approximations of the period [10]. In the case of relativistic oscillators however, this criterion leads to expressions that are not only cumbersome but also divergent at large amplitudes. Fig. 1 shows the relative error of the second order approximation calculated using the method outlined in Ref. [10]. Even though higher order approximations can be computed, the second order calculation exhibits the best accuracy at large amplitudes. Although higher order corrections improve the accuracy near the low energy regime, they also increase the divergence of the solution at large amplitudes.

The alternative criterion given by Eq. (35) of Ref. [10] is also found to be problematic because the resulting ω goes to infinity as $A\to\infty$. Since all terms in Eq. (12) are proportional to powers of $\frac{1}{\omega}$, then any finite order approximation incorrectly goes to zero in the ultrarelativistic limit $A\to\infty$.

We begin the construction of our solution with the correct asymptotic behavior of the period. By noting from Eq. (6) that $\phi(A)-\phi(x)\gg 1$ as $A\to\infty$, it can be shown that Eq. (7) simplifies to what is expected for a photon bouncing in a box of width $2A$:

$$T\sim 4A. \tag{15}$$

This limit will serve as the asymptotic boundary condition that determines our variational parameter.

Proceeding, the zeroth order truncation of the series in Eq. (12) is

$$T=\frac{2\pi}{\omega}. \tag{16}$$

To determine ω in the above equation, we assume the form that directly satisfies Eq. (13), namely

$$\omega=\sqrt{R(\theta)}. \tag{17}$$

Hence ω is still dependent on the variable of integration θ since harmonics of the form $\cos^{2m}\theta$ are present. To remove this, harmonics are treated as free parameters, denoting $\cos^{2m}\theta$ as λ_{2m} for integers $m\geqslant 0$.

Next, observing the asymptotic behavior of ω as $A\to\infty$, it can be shown that for an even potential with leading order x^{2m}, we have

$$\omega\sim\frac{1}{A}\sqrt{\frac{\sum_{n=0}^{m-1}(\lambda_{2n}-\lambda_{2(m+n)})}{1-2\lambda_{2m}+\lambda_{4m}}}. \tag{18}$$

Substituting Eqs. (15) and (18) in the large amplitude limit of Eq. (16) will result in a relation between all orders of free parameters that exist in R. This relation can be used recursively to provide