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Synchronization analysis of singular hybrid coupled networks ☆

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ABSTRACT

In this Letter, singular hybrid coupled systems are introduced to describe complex networks with a special class of constraints. The synchronization problem of singular hybrid coupled systems with time-varying nonlinear perturbation is investigated. A sufficient condition for global synchronization is derived based on the Lyapunov stability theory. The singular system is regular and impulse free. Finally, a numerical example is provided to illustrate the effectiveness of the proposal conditions.

1. Introduction

In the past decades, complex dynamical networks have become a popular research subject and are attracting more and more attention from many fields of scientific research. Examples of complex networks include the World Wide Web, the Internet, electrical power grids, food webs, biological neural networks, telephone cell graphs, coauthorship and citation networks of scientists, etc. The universality of complex networks naturally stimulates the current intensive study of the subject. In the study, one of the basic and significant characteristics is the synchronization of all dynamical nodes in a complex network. In fact, synchronization is a ubiquitous phenomenon in nature. Roughly speaking, if two systems have something in common, then a synchronization may occur between them when they interact. As a result, the synchronization of coupled dynamical networks has become a focal point in researching complex networks [1–7].

It is worthwhile to mention that we usually have to consider some algebraic constraints of complex networks in modeling the real world problems. For instance, communication resources are always limited and required to be allocated to different levels of privileged users. Hence, special constraints are needed in the resource allocation process. Constructing a complex network model with a set of constraints is necessary and indispensable. However, we will encounter two kinds of difficulties:

Q_1) How to construct a complex network model with a set of constraints for a special class of networks?

Q_2) How to investigate those properties such as stability, synchronization and robust stability for a given complex network with constraints?

It is a challenging task to answer the above two questions since it is very difficult to give an explicit expression for complex networks with a set of constraints. In addition, under a set algebraic constraints, one do not know whether the solutions exist, not to mention of the stability or synchronization of the solutions. Therefore, in this Letter, our objective is to investigate a special class of networks and address some interesting issues of the above problems. Inspired by the Letter [8], we introduce the following singular hybrid coupled systems in this Letter

$$E\dot{x}_i(t) = Ax_i(t) + f\left(x_i(t), t\right) + c\sum_{j=1}^{N} b_{ij}\Gamma x_j(t),$$
$$i = 1, 2, \ldots, N, \tag{1}$$

where matrix E may be singular, and $0 < \mathrm{rank}(E) = r < n$. $A \in R^{n\times n}$ is a constant matrix. N is the number of coupled nodes, $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in R^n$ are state variables of node i, $f(x_i(t), t) \in$

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R^n is vector-valued time-varying nonlinear function. The constant $c>0$ denotes the coupling strength, and $\Gamma=\mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_n)\in R^{n\times n}$ is a 0–1 diagonal matrix with specific $\gamma_i=1$ and 0 for others. $B=(b_{ij})_{N\times N}$ denotes the coupling configuration of the entire network. If there is a connection between node i and node j, then $b_{ij}=b_{ji}=1$ $(i\neq j)$; otherwise, $b_{ij}=b_{ji}=0$ $(i\neq j)$. Define the diagonal elements of matrix B such that

$$b_{ii}=-\sum_{j=1,j\neq i}^{N}b_{ij}=-\sum_{j=1,j\neq i}^{N}b_{ji},\quad i=1,2,\ldots,N.$$

Generally, we also call model (1) as the singular complex networks. Without loss of generality, let matrix $E=\begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$ with I_r is a $r\times r$ identity matrix, system (1) can be written

$$\dot{x}_i^{(1)}(t)=A_1x_i^{(1)}(t)+A_2x_i^{(2)}(t)+f_1\big(x_i(t),t\big)+c\sum_{j=1}^{N}b_{ij}\Gamma_1x_j^{(1)}(t),\quad i=1,2,\ldots,N, \tag{2}$$

and

$$0=A_3x_i^{(1)}(t)+A_4x_i^{(2)}(t)+f_2\big(x_i(t),t\big)+c\sum_{j=1}^{N}b_{ij}\Gamma_2x_j^{(2)}(t),\quad i=1,2,\ldots,N, \tag{3}$$

where matrix $A=\begin{pmatrix} A_1 & A_2\\ A_3 & A_4\end{pmatrix}$, $x_i(t)=((x_i^{(1)}(t))^T,(x_i^{(2)}(t))^T)^T$ with $x_i^{(1)}(t)\in R^r, x_i^{(2)}(t)\in R^{n-r}$, $f(x_i(t),t)=(f_1^T(x_i(t),t),f_2^T(x_i(t),t))^T$ with $f_1(x_i(t),t)\in R^r$, $f_2(x_i(t),t)\in R^{n-r}$, and $\Gamma=\mathrm{diag}(\Gamma_1,\Gamma_2)$ with $\Gamma_1\in R^{r\times r}$, $\Gamma_2\in R^{(n-r)\times(n-r)}$.

Obviously, the singular systems (1) are equivalent to models (2) satisfying conditions (3). Here, systems (2) are still considered as hybrid coupled networks and Eqs. (3) are a set algebraic constraints of models (2). Moreover, the constraints in (3) also consist of nonlinear terms. As a result, a kind of complex network models is constructed with a set of algebraic constraints. Hence, we have given one possible answer to question Q_1.

In fact, singular systems give a more general description of physical systems than the normal one (i.e., $E=I$, full rank). And for this reason, many studies have extended concepts and results from the regular systems theory into the realm of singular systems [8–18]. However, most of the authors only discussed the stability of linear singular systems [10,14–18] rather than the nonlinear properties. The reasons are that of nonlinear singular systems is very difficult and complicated when the system regularity and impulse elimination are needed to be considered. In recent years, there are some breakthroughs in nonlinear singular systems. In [8,9], the authors discuss the stability for the following continuous-time singular system with time-varying nonlinear perturbations:

$$E\dot{x}(t)=Ax(t)+f\big(x(t),t\big). \tag{4}$$

It should be noted that, the above model (4) is a special case of system (1).

It is well known that there are tremendous difficulties in investigating some global properties of complex networks due to the inherent complexity issues, not to mention of those of singular complex networks for the system regularity and impulse elimination. Moreover, the introduction of nonlinear perturbation $f(x_i(t),t)$ in model (1) is a general issue to be dealt with in complex networks, and yet it is a more difficult problem in singular complex networks. Therefore, our purpose of this work is to discuss the synchronization of singular hybrid coupled systems (1)—to give an answer to question Q_2.

Notation: Throughout this Letter, I stands for the identity matrix. The superscript "T" represents the transpose. For $\forall x=(x_1,x_2,\ldots,x_n)^T\in R^n$, the notation $\|x\|=(\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. For a matrix A, $\lambda_m(A)$ and $\lambda_M(A)$ denote the minimal and maximal eigenvalues of matrix A respectively. $\|A\|$ denotes the spectral norm defined by $\|A\|=(\lambda_M(A^TA))^{\frac{1}{2}}$. For real symmetric matrices X and Y, $X>Y$ (or $X\geqslant Y$) means that matrix $X-Y$ is positive definite (or positive semidefine).

2. Main results

In the Letter, it is assumed that system (1) is connected in the sense that there are no isolated clusters, i.e., matrix B is an irreducible matrix. With the assumption, we obtain that zero is an eigenvalue of B with multiplicity 1 and all the other eigenvalues of B are strictly negative, which are denoted by $0=\lambda_1>\lambda_2\geqslant\lambda_3\geqslant\cdots\geqslant\lambda_N$.

Definition 1. The singular network in (1) is said to achieve asymptotic synchronization if

$$x_1(t)=x_2(t)=\cdots=x_N(t)=s(t),\quad \text{as } t\to+\infty, \tag{5}$$

where $s(t)$ satisfies the equation $E\dot{s}(t)=As(t)+f(s(t),t)$.

Following [2], we let $\bar{x}(t)=\frac{1}{N}\sum_{i=1}^N x_i(t)$, $e_i(t)=x_i(t)-\bar{x}(t)$, $i=1,2,\ldots,N$. The singular system (1) can be written as

$$E\dot{e}_i(t)=Ae_i(t)+f\big(x_i(t),t\big)-\bar{f}+c\sum_{j=1}^{N}b_{ij}\Gamma e_j(t),\quad i=1,2,\ldots,N, \tag{6}$$

where $\bar{f}=\frac{1}{N}\sum_{l=1}^N f(x_l(t),t)$. Next, the objective is to find a condition to ensure the solutions of singular system (6) are globally asymptotically stable about $e_i(t)=0$, $i=1,2,\ldots,N$.

Model (6) can be written as

$$E\dot{e}(t)=Ae(t)+F\big(e(t),t\big)+c\Gamma e(t)B^T, \tag{7}$$

where $e(t)=(e_1(t),e_2(t),\ldots,e_N(t))$, $F(e(t),t)=(f(x_1(t),t)-\bar{f}, f(x_2(t),t)-\bar{f},\ldots,f(x_N(t),t)-\bar{f})$. Since B is a symmetric matrix, there exists a unitary matrix $w=(w_1,w_2,\ldots,w_N)\in R^{N\times N}$ such that $B=w\Lambda w^T$ with $ww^T=I$ and $\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_N)$. Let $y(t)=e(t)w$, along with (7), one has

$$E\dot{y}(t)=Ay(t)+F\big(e(t),t\big)w+c\Gamma y(t)\Lambda, \tag{8}$$

where $y(t)=(y_1(t),y_2(t),\ldots,y_N(t))$, $y_i(t)=e(t)w_i\in R^n$, $i=1,2,\ldots,N$. Then, model (8) can be rewritten

$$\begin{aligned}E\dot{y}_i(t)&=Ay_i(t)+F\big(e(t),t\big)w_i+c\lambda_i\Gamma y_i(t)\\&=(A+c\lambda_i\Gamma)y_i(t)+F\big(e(t),t\big)w_i.\end{aligned} \tag{9}$$

Assumption 1. Assume that there exist nonnegative constants L_{ij} such that $\|f(x_i(t),t)-f(x_j(t),t)\|\leqslant L_{ij}\|x_j(t)-x_i(t)\|$, $i\neq j, i,j=1,2,\ldots,N$.

Assumption 2. There exist matrices P_i such that

$$E^TP_i=P_i^TE\geqslant 0,\quad i=1,2,\ldots,N, \tag{10}$$

and

$$A^TP_1+P_1^TA<0,\qquad (A+c\lambda_i\Gamma)^TP_i+P_i^T(A+c\lambda_i\Gamma)\leqslant-\eta_iI,\quad i=2,3,\ldots,N, \tag{11}$$

where $\eta_i>\frac{4L}{N}(N-1)\|P_i\|$, $L=\sum_{i=1}^N L_i$, $L_i=\sum_{l=1,l\neq i}^N L_{il}$.