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Construction of bound entangled edge states with special ranks

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Abstract

Relating to a conjecture on the decomposability of positive maps in $3 \otimes 3$, we solve the open problem of the existence of $(5,5)$ and $(6,6)$ PPT edge states.

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1. Introduction

The power of entanglement as a physical resource in quantum information and computation has motivated a wide scale study in the mathematical structure of entanglement. The breakthrough came when the Horodecki family [1] linked the question of separability to the classification of positive maps in matrix algebras. In the 1960–1980s important progress was made in the classification of positive maps. A paper by Choi [2] in 1982 reviews the main results until then and basically contains the skeleton of present entanglement theory.

Let M_n stand for the set of all $n \times n$ complex matrices. Probably the most important result in the theory of positive maps is that every positive map between M_2 and M_n for $n \leqslant 3$ is decomposable. A decomposable map can be decomposed as the sum of a completely positive map (CPM) and the combination of transposition and a CPM. In entanglement theory this translates to the fact that for all states ρ in $M_n \otimes M_m$ $(nm \leqslant 6)$ positivity of the partial transposition

$$(\mathbb{1} \otimes T)\rho = \rho^{T_B} \geqslant 0$$

is a necessary and sufficient condition for separability. For higher dimensions this is not the case and there exist entangled states with a positive partial transposition (PPTES). From a mathematical point of view the structure of PPTES in $M_2 \otimes M_4$ and $M_3 \otimes M_3$ are therefore of great interest. In the present Letter we are concerned with the latter. Only a handful of examples are available in this dimension:

(A) The Størmer matrix [3,4].
(B) The Choi matrix [2,5].
(C) The 7-parameter chessboard states [6].
(D) The 6-parameter UPB states in [7] and neighbourhood [8,9].
(E) The Horodecki matrix [10].
(F) The Ha et al. matrices [11–13]. In matrix structure, these matrices lie in between (A) and (B).

Construction of PPTES is a non-trivial task, and the UPB construction is really the only known automatic procedure [14]. The other constructions are very much trial and error and in the spirit of Pólya's traditional mathematics professor 'In order to solve this differential equation you look at it till a solution occurs to you' [15]. Yet, given a PPTES there are several tools available to show it is entangled:

(i) A first one is the so-called realignment criterion [16–18] which just like the partial transposition reorders matrix entries. Here entanglement is guaranteed when the trace norm of the realigned density matrix is larger than one.

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(ii) A second option is making use of non-decomposable positive maps, or alternatively non-decomposable entanglement witness [19]. This method lacks the operational character of the realignment criterion, as it is non-trivial to prove the positiveness of a map. However, we know that every PPTES can be detected by some entanglement witness and hence this criterion is a much stronger one than the realignment criterion.

(iii) In Ref. [20] Doherty et al. used entanglement witnesses to devise a computational algorithm which detects all entangled states. Furthermore for a given entangled state their algorithm outputs an entanglement witnesses W detecting that state. This operator W can always be written in a k-SOS (sum of squares) form which makes it easy to prove analytically that it is indeed an entanglement witness (see the original reference for details).

(iv) The range criterion offers a remarkable simple criterion for PPTES with small rank [10,21]. It dictates that for a state ρ to be separable there must exist a set of product vectors $\{|a_i\rangle|b_i\rangle\}$ spanning the range of ρ such that $\{|a_i\rangle|b_i^*\rangle\}$ span the range of ρ_B^T. In particular, we say that a state ρ strongly violates the range criterion if there is no product vector $|a_i\rangle|b_i\rangle$ in the range of ρ such that $|a_i\rangle|b_i^*\rangle$ is in the range of ρ^{T_B}. A state which strongly violates the range criterion will be called an edge state in view of the following theorem.

Theorem 1. (Lewenstein et al. [22–27]) *(i) A PPTES δ is an edge state if and only if for all $\epsilon \geqslant 0$ and all separable $|ab\rangle$,*

$$\delta - \epsilon|ab\rangle\langle ab|$$

is not positive or does not have a positive partial transpose.

(ii) Every PPTES ρ can be decomposed as

$$\rho = (1-p)\rho_{\text{sep}} + p\delta,$$

with ρ_{sep} a separable state and δ an edge state.

This theorem implies that knowledge of edge states is sufficient to characterise PPTES.

In Ref. [28] a study of the Schmidt number of density matrices was made. The Schmidt number [29,30] of a density matrix is defined as the minimum—over all convex decompositions of a density matrix into pure states, of the maximum Schmidt rank in such a decomposition. In particular they conjectured that all bound entangled states in $M_3 \otimes M_3$ have Schmidt number two. This can also be casted in the language of 2-positive maps [13]. A 2-positive map is a linear map Λ such that $\mathbb{1} \otimes \Lambda(\rho) \geqslant 0$ for all Schmidt number two states ρ. It follows that the conjecture is equivalent to the statement that every 2-positive map between 3×3 is positive on PPTES states. To prove this, it is sufficient to prove it for edge states. In Ref. [28] a proof was presented for edge states of rank 4. Denoting the rank of ρ by N and the rank of ρ^{T_B} by M, they then analysed the situation for edge states with different ranks (N, M). Unfortunately at the time they did their analysis, only edge states of dimension $(4,4)$ and $(6,7)$ were known. Recently, in a very interesting paper [13], Ha and Kye found edge states for all ranks except $(5,5)$ and $(6,6)$. The main result of this Letter is the construction of edge states with rank $(5,5)$ and rank $(6,6)$, respectively (Sections 2 and 3). In the final section we show that our states do not seem to contradict the original conjecture. For a generalisation of the conjecture to higher-dimensional systems the reader is referred to [31].

2. A (5, 5) edge PPTES

Consider the following $(5,5)$ state:

$$\rho_{(5,5)} = \frac{1}{13}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & -2 & 3 \end{bmatrix},$$

and its partial transpose:

$$\rho_{(5,5)}^{T_B} = \frac{1}{13}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & -2 & 3 \end{bmatrix}.$$

It is not so hard to verify that both operators are positive semi-definite and have rank 5. An analytical expression of their eigenvectors and eigenvalues is however quite complex. To show that $\rho_{(5,5)}$ is an edge state we will show that it violates the strong range criterion. For this we will use the 'divide and conquer technique' from [10].

It is not so hard to see that every vector in the range of $\rho_{(5,5)}$ can be written in the form

$$V = (0, A, -E-F, C, 0, D, D, E, F), \quad A, C, D, E, F \in \mathbb{C}.$$

Now we have look at those vectors which can be written as a product

$$V = (s, t, v) \otimes (x, y, z) = (sx, sy, sz, tx, ty, tz, vx, vy, vz).$$

Taking these two conditions together we can therefore characterise all product vectors in the range of $\rho_{(5,5)}$.

From the condition $sx = 0$ we can distinguish the following sub cases:

1. $x = 0$, $s \neq 0$, we have $vx = D = 0 = tz$ and therefore either $t = 0$ or $z = 0$. Without loss of generality we can also put $s = 1$.

1.1. $t = 0$, as $v(y+z) = -z$ we have $v = -z/(y+z)$. When $y = -z$, then $z = 0 = -y = x$ and we obtain the null vector. Thus the only case that remains is

$$V = s\left(1, 0, \frac{-z}{y+z}\right) \otimes (0, y, z),$$

with $y \neq -z$.