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On the chaotic aspects of three wave interaction in a magnetized plasma

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ABSTRACT

Nonlinear processes in magnetized plasma are very much important for the proper understanding of many space and astrophysical events. One of the most important type of study has been done in the domain of Alfven waves. Here we show that a Galerkin type approximation of the DNLS (Derivative Nonlinear Schrödinger) equation describing such wave propagation leads to a new type of nonlinear dynamical systems, very much rich in chaotic properties. Starting with the detailed analysis of fixed points and stability zones we make an in depth study of the unstable periodic orbits, which span the whole attractor. Next the birth of a Hopf bifurcation is identified and normal form, limit cycle analyzed. In the course of our study the detailed structure of the attractor is analyzed. A possibility of internal crisis is also indicated. These results will help in the choice of the plasma parameters for the actual physical situation.

1. Introduction

Of late rigorous research activities have been seen to investigate collective plasma modes in classical and quantum plasmas [1–6]. The inherent nonlinearity of such processes makes it imperative that a detailed stability analysis is undertaken. One of the most important class of problems occur in the domain of magnetized plasma, where the analysis of Alfven waves plays a leading role. In particular it may be mentioned that the coupling of Alfven–Langmuir–Whistler waves [7–9] plays a significant role in planetary magnetosphere. The nonlinear interaction of such waves is still a very important [10–12] one, and requires a very detailed study [13–16]. A recent paper by Brodin et al. [17] has studied the three waves process in a cold magnetized plasma. Also it has been seen that radio energy bursts from sun can be produced via a nonlinear conversion of Langmuir waves into high frequency electromagnetic cyclotron pulse, through coupling with similar low frequency waves. On the other hand, a kinetic Alfven and Whistler wave theory have been proposed by Voitenko and Goossens [15] in this respect. So it is very apparent that a sort of three wave interaction process plays a central role in these processes. Here in this communication we have considered a Galerkin type approximation [18] of the DNLS equation describing the dynamics of a large amplitude nonlinear Alfven wave propagating along the ambient magnetic field, to obtain a new set of nonlinear ordinary differential equations (ode's), describing the three wave process. As this set of equations is highly nonlinear it is very much important to investigate those set of parameter values which lead to stable or unstable mode [19–22]. Below we show how the instability is built up and through the formation of unstable periodic orbits span the whole chaotic attractor. In this connection, a new branch of instability is also identified leading to Hopf bifurcation [23–26], whose detailed structure is exposed via normal form analysis. To start with we analyze the origin and stability of various fixed points by the Routh–Hurwitz criterion. Here we observe the transition to a Hopf bifurcation channel later explored with the help of normal form. The form of limit cycle is also obtained. Later a detailed investigation is undertaken to explore the population of unstable periodic orbits of the attractor by a modified approach of Newton–Raphson method for flow [27]. Lastly we have used MATCONT [28] to study the unfolding of bifurcation, with its transition to Hopf state which shows agreement with the detailed numerical results.

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2. Formulation

The dynamics of large amplitude Alfven waves propagating along the ambient magnetic field can be described by the DNLS equation,

$$\left(i\frac{\partial}{\partial t}-\hat{\gamma}\right)B+i\alpha\frac{\partial}{\partial x}\left(B|B|^2\right)+\beta\frac{\partial^2 B}{\partial x^2}=0 \tag{1}$$

here $\hat{\gamma}$ is the linear growth or damping, α stands for the sign of the nonlinearity and β that of dispersion.

The three term Galerkin approximation is done by writing B as

$$B=\sum_{i=1}^{3}B_i(t)\exp\left\{-i(k_i x-\omega_i t)\right\} \tag{2}$$

where we assume $2k_1=k_2+k_3$. From the linear dispersion relation we have $\omega_i=-\beta k_i^2$ $(i=1,2,3)$ and $\omega_{2,3}-\omega_1=\Delta'_{1,2}$ is the phase difference. We furthermore represent each complex amplitude $B_i(t)$ as

$$B_i(t)=R_i(t)\exp\left\{i\theta_i(t)\right\} \tag{3}$$

$i=1,2,3$, while $(R_i,\theta_i(t))$ are real. To simplify the resulting dynamical system equations we impose some conditions on the various quantities,

$$R_1=R_2,\qquad r_1=r_2,\qquad \alpha=\beta=-1,$$

alongwith $T=\gamma_0 t$ and $k_1\approx k_2\approx k_3$,

$$a_1^2=\frac{k_1}{\gamma_0}R_1^2,\qquad a_2^2=\frac{k_1}{\gamma_0}R_2^2,\qquad \gamma=-\frac{\gamma_1}{\gamma_0},$$
$$\phi=-\psi,\qquad \delta=-\frac{\Delta'}{\gamma_0} \tag{4}$$

where $\Delta'=(\Delta'_1+\Delta'_2)/2$. So that one gets

$$\dot{a}_1=a_1+2a_1a_2^2\sin\phi,$$
$$\dot{a}_2=-\gamma a_2-a_1^2a_2\sin\phi,$$
$$\dot{\phi}=-2\delta+2\left(a_2^2-a_1^2\right)+2\left(2a_2^2-a_1^2\right)\cos\phi \tag{5}$$

where $\dot{a}_1$, etc., denote da_1/dT and

$$\psi=2\theta_1-\theta_2-\theta_3-2\Delta' t.$$

Eq. (5) describes the three wave process under consideration.

3. Stability analysis

To start with we observe that the fixed point of Eq. (5) is given as

$$a_0^*=\sqrt{\frac{\gamma}{D}};\qquad a_1^*=\frac{1}{\sqrt{2D}};\qquad \theta^*=-\sin^{-1}(D),$$

where D is given as

$$D=\frac{(\gamma-1)\sqrt{4\delta^2-4\gamma+3}-\delta(2\gamma-1)}{2\{\delta^2+(\gamma-1)^2\}}$$

provided $4\delta^2-4\gamma+3>0$. Furthermore we fix $\delta=-6$ and vary γ as the bifurcation parameter. The characteristic equation correspondence to this is

$$f(\lambda)=\lambda^3+e_1\lambda^2+e_2\lambda+e_3=0 \tag{6}$$

where

$$e_1=2(\gamma-1),$$
$$e_2=\frac{2\gamma}{D^2}\left(2D^2+4-3\xi\right),$$
$$e_3=\frac{8\gamma}{D}\left\{(\gamma-1)D-6\xi\right\}$$

with $\xi=\frac{-12D+2\gamma-1}{2(\gamma-1)}$. By the Routh–Hurwitz criterion the real part of the roots of Eq. (6) are negative if and only if

$$e_1>0,\qquad e_3>0,\qquad e_1e_2>e_3$$

leading to the restriction $1<\gamma<1.3215$. Note that all the coefficients of Eq. (6) are positive for $\gamma>1$. Consequently the instability arises if there is two complex conjugate zeros, say $\lambda_1=i\Omega$, $\lambda_2=-i\Omega$, then $\lambda_3=-2(\gamma-1)$ which is on the boundary of the stability and we obtain the critical value $\gamma=\gamma_0=1.3215$ and $\lambda_3=-0.643<0$. This set of complex eigenvalues leads to Hopf bifurcation, which we take up in a later section. But here we study the general structure of the unstable periodic orbits (UPO's) which populate the attractor.

Before going to the actual results we briefly describe the methodology adopted for getting information about the UPO's. Though basically it is a variant of the Newton's method for finding the fixed points of a Poincaré map, yet some details are given here for the sake of completeness.

Consider a dynamical system,

$$\dot{x}=f(x) \tag{7}$$

where $x=\{x_1,x_2,\ldots,x_d\}$ is a d-dimensional vector and $f=\{f_1(x),f_2(x),\ldots,f_d(x)\}$. Let $X_t(X_0)$ be the flow of (7) for a given initial condition X_0, that is the value of the orbit of X_0 at time t. Let $t=\sigma$ be the time of a trajectory started in X_0 takes to complete one iteration of some Poincaré map, so

$$X_1=X_\sigma(X_0)$$

is the next point in the map iteration. The fixed point is a solution of

$$X_\sigma(\bar{X})=\bar{X}.$$

By Newton's approach we set

$$\bar{X}=X_0+\Delta X$$

with initial guess X_0 close to the desired point and an initial guess τ_0 for the return time of $\bar{X}$

$$\sigma=\tau_0+\Delta\tau$$

then Taylor expansion of X yields

$$X_\sigma(\bar{X})=X_{\tau_0}(X_0)+\frac{\partial X_{\tau_0}(X_0)}{\partial t}\Delta\tau+D_X X_{\tau_0}(X_0)\Delta X,$$

where $\frac{\partial X_{\tau_0}(X_0)}{\partial t}=f(X_{\tau_0}(X_0))$. Now using $X_\tau(\bar{X})=\bar{X}=X_0+\Delta X$, one gets

$$(I-J)\Delta X-f(X_1)\Delta\tau=X_1-X_0, \tag{8}$$

where I is the identity matrix, J is the Jacobian to be obtained by integrating the variational equation

$$\dot{J}=D_X f(X_0)J \tag{9}$$

along with Eq. (7), with $J=I$ as the initial condition. The solution X_1 of Eq. (7) at each iteration are not necessarily on the Poincaré section unless a stroboscopic section is used. A possible method to restrict this is to assume that Poincaré section be given as

$$(X_1-X_0)a=0 \tag{10}$$

where a is a vector normal to the plane, X_0 is on it. Adding (8) to (10) guarantees that X_1 is always on the plane when we iterate

$$\begin{pmatrix} I-J & -f(X_1)\\ a & 0\end{pmatrix}\begin{pmatrix}\Delta X\\ \Delta\tau\end{pmatrix}=\begin{pmatrix}X_1-X_0\\ 0\end{pmatrix}. \tag{11}$$

In the present case, we have two bifurcation parameters (γ and δ). In Fig. 1(a) we have fixed δ to -6.015 and shown the bifurcation