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The homotopy analysis method for Cauchy reaction–diffusion problems

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Abstract

In this Letter, the homotopy analysis method (HAM) is employed to obtain a family of series solutions of the time-dependent reaction–diffusion problems. HAM provides a convenient way of controlling the convergence region and rate of the series solution.

Keywords: Homotopy analysis method; Reaction–diffusion equation

1. Introduction

The one-dimensional, time-dependent reaction–diffusion equation of interest in this Letter is of the following form:

$$\frac{\partial w}{\partial t}(x,t) = D\frac{\partial^2 w}{\partial x^2}(x,t) + p(x,t)w(x,t),$$
$$(x,t) \in \Omega \subset \Re^2, \tag{1}$$

where w is the concentration, p is the reaction parameter and $D > 0$ is the diffusion coefficient. The initial and boundary conditions are

$$w(x,0) = g(x), \quad x \in \Re \tag{2}$$

$$w(0,t) = f_0(t), \qquad \frac{\partial w}{\partial x}(0,t) = f_1(t), \quad t \in \Re. \tag{3}$$

Reaction–diffusion equations describe a wide variety of nonlinear systems in physics, chemistry, ecology, biology and engineering [1–4].

Approximate series solutions of the reaction–diffusion problems were given by Lesnic [5] using the analytic Adomian decomposition method (ADM) and by Dehghan and Shakeri [6] using the variational iteration method (VIM). A more general analytic method is the homotopy analysis method (HAM), first proposed by Liao in his Ph.D. thesis [7]. A systematic and clear exposition on HAM is given in [8]. In recent years, HAM has been successfully employed to solve many types of linear and nonlinear problems in science and engineering [9–32]. HAM contains a certain auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called h-curve, it is easy to find the valid regions of h to gain a convergent series solution. Thus, through HAM, explicit analytic solutions of nonlinear problems are possible.

In this Letter, the Cauchy reaction–diffusion equations shall be solved by HAM. It is demonstrated that the solutions obtained by ADM are special cases of the HAM solutions.

2. Basic ideas of HAM

We consider the following differential equation,

$$N\big[w(x,t)\big] = 0,$$

where N is a nonlinear operator, x and t denotes the independent variables, $w(x,t)$ is an unknown function, respectively. By means of generalizing the traditional homotopy method, Liao [8] constructs the so-called *zero-order deformation equation*

$$(1-q)L\big[\phi(x,t;q) - w_0(t)\big] = q\hbar H(x,t)N\big[\phi(x,t;q)\big], \tag{4}$$

where $q \in [0,1]$ is an embedding parameter, h is a nonzero auxiliary function, L is an auxiliary linear operator, $w_0(x,t)$ is an initial guess of $w(x,t)$ and $\phi(x,t;q)$ is an unknown function. It

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is important to note that, one has great freedom to choose auxiliary objects such as h and L in HAM. Obviously, when $q=0$ and $q=1$, both

$$\phi(x,t;0)=w_0(x,t) \quad \text{and} \quad \phi(x,t;1)=w(x,t),$$

hold. Thus as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $w_0(x,t)$ to the solution $w(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q, one has

$$\phi(x,t;q)=w_0(x,t)+\sum_{m=1}^{+\infty} w_m(x,t)q^m, \tag{5}$$

where

$$w_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{6}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, then the series (5) converges at $q=1$ and

$$\phi(x,t;1)=w_0(x,t)+\sum_{m=1}^{+\infty} w_m(x,t),$$

which must be one of solutions of the original nonlinear equation, as proved by Liao [8]. As $\hbar=-1$ and $H(x,t)=1$, Eq. (4) becomes

$$(1-q)L\big[\phi(x,t;q)-w_0(x,t)\big]+qN\big[\phi(x,t;q)\big]=0, \tag{7}$$

which is used mostly in the homotopy-perturbation method [33].

According to (6), the governing equation can be deduced from the *zero-order deformation equation* (4). Define the vector

$$\vec{w}_n=\big\{w_0(x,t),\ w_1(x,t),\dots,w_n(x,t)\big\}.$$

Differentiating (4) m times with respect to the embedding parameter q and then setting $q=0$ and finally dividing them by $m!$, we have the so-called *mth-order deformation equation*

$$L\big[w_m(x,t)-\chi_m w_{m-1}(x,t)\big]=\hbar H(t)R_m(\vec{w}_{m-1}), \tag{8}$$

where

$$R_m(\vec{w}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{9}$$

and

$$\chi_m=\begin{cases}0, & m\leqslant 1,\\ 1, & m>1.\end{cases}$$

It should be emphasized that $w_m(x,t)$ $(m\geqslant 1)$ is governed by the linear equation (8) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation softwares such as Maple and Mathematica.

3. HAM solutions

3.1. Solutions using initial condition

In order to solve Eqs. (1) and (2) (i.e., partial t-solution) by HAM, we choose the initial approximation

$$w_0(x,t)=g(x), \tag{10}$$

and the auxiliary linear operator

$$L\big[\Phi(x,t;q)\big]=\frac{\partial\phi(x,t;q)}{\partial t}, \tag{11}$$

with the property

$$L[c_1]=0,$$

where c_1 is an integral constant. Furthermore, Eq. (1) suggests that we define the nonlinear operator as

$$N\big[\phi(x,t;q)\big]=\frac{\partial\phi(x,t;q)}{\partial t}-D\frac{\partial^2\phi(x,t;q)}{\partial x^2}+p(x,t)\phi(x,t;q). \tag{12}$$

Using the above definition, we construct the *zeroth-order deformation equation*

$$(1-q)L\big[\phi(x,t;q)-w_0(x,t)\big]=q\hbar N\big[\phi(x,t;q)\big], \tag{13}$$

and the *mth-order deformation equation*

$$L\big[y_m(x,t)-\chi_m w_{m-1}(x,t)\big]=\hbar R_m(\vec{w}_{m-1}), \tag{14}$$

with the initial condition

$$w_m(x,0)=0, \tag{15}$$

where

$$R_m(\vec{y}_{m-1})=(w_{m-1})_t-D(w_{m-1})_{xx}+pw_{m-1}. \tag{16}$$

Now, the solution of the mth-order deformation Eq. (14) for $m\geqslant 1$ becomes

$$w_m(x,t)=\chi_m w_{m-1}(x,t)+\hbar\int_0^t R_m(\vec{w}_{m-1})\,\mathrm{d}\tau+c_1, \tag{17}$$

where the integration constants c_1 is determined by the initial conditions (15).

3.2. Solutions using boundary conditions

In this section to solve Eqs. (1) and (3) (i.e., partial x-solution) by HAM, we choose the initial boundary approximation

$$w_0(x,t)=f_0(t)+xf_1(t), \tag{18}$$

and the auxiliary linear operator

$$L\big[\Phi(x,t;q)\big]=\frac{\partial^2\phi(x,t;q)}{\partial x^2}, \tag{19}$$

with the property

$$L[c_2+xc_3]=0,$$