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# Fidelity thresholds in single copy entanglement distillation

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**Abstract**

Various aspects of distillation of noisy entanglement and some associated effects in quantum error correction are considered. In particular, we prove that if only one-way classical communication (from Alice to Bob) is allowed and the shared $d \otimes d$ state is not pure then there is a threshold for optimal entanglement fraction $F$ of the state that can be obtained in single copy distillation process. This implies that to get (probabilistically) arbitrary good conclusive teleportation via mixed state at least one classical bit of backward communication (from Bob to Alice) has to be sent. We provide several other threshold properties in this context including in particular the existence of ultimate threshold of optimal $F$ for states of full rank. Finally the threshold results are discussed in context of probabilistic error correction.


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## 1. Introduction

Quantum teleportation is one of the most interesting phenomenon in quantum information theory [1]. In its original version it involves perfect transmission of unknown spin $s$ state $\phi$ from Alice to Bob by means of shared maximally entangled pair (serving as a quantum channel) of two spin $s$ particles and $2s + 1$ classical bits sent from Alice to Bob via some classical channel. First step of the scheme is due to Alice. It involves the joint complete von Neumann measurement of observable with maximally entangled vectors Alice performs on both particle $\phi$ and one member of maximally entangled pair. After getting the result Alice communicates it to Bob who performs some unitary operation on the second member of the pair reconstructing the state $\phi$ though, in general, neither Alice nor Bob know the state $\phi$ which is teleported. However, more general schemes involving mixed states [2], conclusive teleportation [3] or general LOCC operations [4] are known. In particular, it has been shown [4] that optimization of teleportation fidelity $f$ under local operation and classical communications is in one to one correspondence with optimization of the so singlet fraction $F$.

On the other hand a natural question arose: *how big $F$ Alice and Bob can achieve given a single copy of strictly mixed quantum state?* It has been shown that for some highly mixed states there exists an unconditional threshold value $F_{\text{threshold}} < 1$ (see [5,6]). However, it turned out that there are some strictly mixed states for which Alice and Bob can get as good $F$ as they want if they use general LOCC with two-way classical communication in a conclusive way [4,11]. In fact, they can achieve $F$ arbitrary close to unity but with probability $p(F)$ going to zero with $F$ approaching unity. This type of conclusive process is called *quasi-distillation*. This leads to a surprising result that sometimes we can achieve arbitrary good teleportation via mixed states (with the probability of the process depending on the fidelity we require).

Now question is whether it is possible to do the trick better, namely, involving only one-way classical communication from Alice to Bob. In the present Letter we show that it is impossible—in this case the threshold value of optimal $F$ for any mixed state is unconditional.

It follows, in particular, that there is a threshold on transmission fidelity $f$ of conclusive teleportation [11] via an unknown

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state from Alice to Bob, i.e., $f \leqslant f_{\text{threshold}} < 1$. It should be stressed that the fact was not obvious because, as it was recalled above, it is not true for some mixed states if two-way communication is allowed. Further we show that such threshold result is true in general (i.e., also if two-way communication is allowed) when the state shared by Alice and Bob is either of full rank or it is supposed to subject only to trace-preserving LOCC operations.

## 2. Improving singlet fraction of single copy: general concepts and methods

### 2.1. Achieving maximal entanglement

Here the aim of the process is to achieve the maximal entanglement which is admitted by the 'physics' of the system. Consider the state described on the Hilbert space $\mathcal{H}$ of the total system, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with dimensions $d_A \equiv \dim \mathcal{H}_A$, $d_B \equiv \dim \mathcal{H}_B$. Such systems are called *$d_A \otimes d_B$ systems*.

For them the family of maximally entangled states has some representative. This is the symmetric state

$$P_+ = |\Psi_+\rangle\langle\Psi_+|,$$
$$|\Psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle, \quad d = \min[d_A, d_B]. \tag{1}$$

Indeed, any other maximally entangled state $P_{\max}$ of $d_A \otimes d_B$ system can be generated from (1) by the equation: $P_{\max} = U_1 \otimes U_2 P_+ U_1^\dagger \otimes U_2^\dagger$ for some unitary operations $U_1$, $U_2$.

Singlet fraction [4] of the state $\varrho$ is defined as follows:

$$F(\varrho) = \langle\Psi_+|\varrho|\Psi_+\rangle. \tag{2}$$

Let us recall that Alice and Bob are allowed to perform *LOCC operations*: the ones involving arbitrary local operations (LO) as well as classical communication (CC). If given operation $\Lambda$ is LOCC we shall write $\Lambda \in$ LOCC. We shall consider here two classes of LOCC operations:

(i) *unilocal* (or *one-way*) where only one party is supposed to communicate the other, here we have two possibilities $A \to B$ (Alice is allowed to call Bob) or $B \to A$ (the opposite case),
(ii) *bilocal* (or *two-way*) if two parties can communicate with each other.

For any of the above actions we have two possibilities. Namely, any $\Lambda \in$ LOCC can be

(i′) *trace-preserving*, i.e., such that for any state $\varrho$ one has $\mathrm{Tr}(\Lambda(\varrho)) = 1$. Alice and Bob always have to keep their particles after performing their actions,
(ii′) *conclusive* (see [3]). Here $\mathrm{Tr}(\Lambda(\varrho)) < 1$. In this case Alice and Bob sometimes throw away the particles if the result of their action is not satisfactory.

We have also the following possibility of single copy distillation. Suppose that Alice and Bob share only *one* copy of $d \otimes d$ state. Then we have two ways of achieving maximal entanglement:

(a) *single copy distillation* (SCD) under a chosen class of operations $\mathcal{L} \subset$ LOCC takes place iff Alice and Bob can perform the action
$$\varrho \xrightarrow[p\text{-probab.}]{\Lambda \in \mathcal{L}} P_+ \tag{3}$$
with probability of success $p = \mathrm{Tr}(\Lambda(\varrho)) > 0$ (for trace-preserving protocol we require $p = 1$),
(b) *single copy quasi-distillation* (SCQD) under a chosen class of operations $\mathcal{L} \subset$ LOCC takes place if there exists a sequence of $\Lambda_n \in \mathcal{L}$ such that
$$\varrho \xrightarrow[p_n\text{-probab.}]{\Lambda_n \in \mathcal{L}} \varrho_n \quad \text{with } F(\varrho_n) \to 1,\ p_n \to 0. \tag{4}$$

**Remark 1.** If $\varrho$ is SCD then it is also SCQD.

Let us recall that the operation is called *separable* iff its action on $\rho$ is of the form

$$\varrho \to \varrho' \equiv \frac{\sum_i A_i \otimes B_i \varrho A_i^\dagger \otimes B_i^\dagger}{\mathrm{Tr}(\sum_i A_i \otimes B_i \varrho A_i^\dagger \otimes B_i^\dagger)}. \tag{5}$$

In particular, any LOCC operation is separable but not vice versa [7].

Some time ago the complete characterization of both SCD and SCQD processes in two qubit case has been provided with help of the Lorentz transformations technique [8]. Below we shall address general questions independent of the (finite) dimension of involved Hilbert space.

### 2.2. Achieving entanglement of smaller Schmidt rank

The Schmidt rank (SR) of the bipartite (i.e., $d_A \otimes d_B$) pure state is the rank of its reduced density matrices (see [9,10] for generalization of SR to mixed states domain) equal to the number of its nonzero eigenvalues. In previous Section 2.1 we have consider methods of achieving the state of maximal entanglement admitted by the physics of the system. But given the $d_A \otimes d_B$ system there is a possibility of more modest objective: to get pure state which is maximally entangled under the constraint of SR bounded by some $m \leqslant d = \min[d_A, d_B]$. Family of such states has its representative of the form

$$P_+^m = |\Psi_+^m\rangle\langle\Psi_+^m|, \quad |\Psi_+^m\rangle = \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} |i\rangle \otimes |i\rangle, \tag{6}$$

with the corresponding $m$-singlet fraction:

$$F^m(\varrho) = \langle\Psi_+^m|\varrho|\Psi_+^m\rangle. \tag{7}$$

We will omit the superscript if $m$ is maximal ($m = d$) All the classification from the previous subsection apply i.e., we have the corresponding so-called *$m \otimes m$ SCD and SCQD processes*. There is a result (see [4]):