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Physics Letters A 354 (2006) 126–136
PHYSICS LETTERS A
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Stability switches, Hopf bifurcation and chaos of a neuron model with delay-dependent parameters

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Received 2 March 2005; received in revised form 12 January 2006; accepted 13 January 2006

Available online 24 January 2006

Communicated by A.P. Fordy

Abstract

It is very common that neural network systems usually involve time delays since the transmission of information between neurons is not instantaneous. Because memory intensity of the biological neuron usually depends on time history, some of the parameters may be delay dependent. Yet, little attention has been paid to the dynamics of such systems. In this Letter, a detailed analysis on the stability switches, Hopf bifurcation and chaos of a neuron model with delay-dependent parameters is given. Moreover, the direction and the stability of the bifurcating periodic solutions are obtained by the normal form theory and the center manifold theorem. It shows that the dynamics of the neuron model with delay-dependent parameters is quite different from that of systems with delay-independent parameters only.

PACS: 05.45.-a; 84.35.+i; 47.20.Ky; 87.19.La

Keywords: Infinite-dimensional system; Delay-dependent parameters; Stability switches; Bifurcation; Center-manifold; Normal form

1. Introduction

Over the past decades, there has been an increasing interest in the study of neuron systems such as in the study of their mathematical modeling and artificial representations. Neural networks are complex and large-scale nonlinear dynamical systems and inevitably they incorporate time delays since the transmission of information between the neurons is not instantaneous. Neural networks with time delays are infinite-dimensional dynamical systems and have richer and more complicated dynamical behavior [1–6]. For simplicity, some researchers studied the dynamical behavior of simple neural network systems with a few number of neurons, and most of papers are devoted to the stability of equilibrium, existence and stability of periodic solutions and chaos of delayed neural network models [7–15].

Recently, a new kind of simple neuron models with delays is proposed. For example, Gopalsamy and Leung [16] studied a delayed neuron model with dynamical threshold effect as follows

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + af\left[x(t) - b\int_{-\infty}^{t} F(t-s)x(s)\,\mathrm{d}s - c\right], \quad (1)$$

where $x(t)$ denotes the neuron response, a denotes the range of the continuous variable, b can be considered as a measure of the inhibitory influence from the past history, c is a off-set constant, s is time delay which denotes the response time of an action. The term $x(t)$ in the argument of function $f(\cdot)$ in Eq. (1) denotes self excitations. $F:[0,+\infty) \to [0,+\infty)$ is a continuous delay kernel function.

On the basis of Lyapunov's method, Gopalsamy and Leung [16] obtained some necessary and sufficient conditions for the existence of globally asymptotically stable equilibrium of Eq. (1). Liao et al. discussed stability and oscillation of this

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doi:10.1016/j.physleta.2006.01.034

model with the weak kernel $F(s) = \alpha \exp(-\alpha s)$ [17] and with strong kernel $F(s) = \alpha^2 s \exp(-\alpha s)$ [18].

In particular, if the kernel functions is a Dirac delta function of the form

$$F(s) = \delta(s - \tau), \quad \tau > 0, \tag{2}$$

then system (1) is changed into the following model with a discrete delay τ

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + af\big[x(t) - bx(t-\tau) - c\big]. \tag{3}$$

The asymptotic behavior and delay-induced oscillations of neuron system (3) are studied by Pakdaman and Malta [19]. Moreover, Ruan et al. [20] gave a detailed analysis of stability and Hopf bifurcation of this model by means of the Lyapunov functional approach. Chaotic behavior of this model with non-monotonously increasing activation function has been discussed by Liao [21].

All of the aforementioned studies on Eq. (3) suppose that the system parameters are constants independent of time delay. However, memory performance of the biological neuron usually depends on time history, its memory intensity is usually lower and lower as time is gradually far away from the current time. That is to say, the parameters in neural networks depend usually on time delay. Compared with the intensive studies on the neural networks with delay-independent parameters, little progress has been achieved for the systems that have delay-dependent parameters. The aim of this Letter is to analyze the dynamics of neuron model (3) with parameter b depending on time delay τ. Special attention is paid to the effect of time delay on the system dynamics including stability, delay-induced oscillations and chaos.

The Letter is organized as follows. In Section 2, the linear stability analysis of system is firstly given on the basis of stability switches, and stability criteria for the trivial solution are presented. In Section 3, an analysis on the steady-state bifurcation, on Hopf bifurcation including the direction and stability of bifurcation periodic solutions of this model is made. The analysis is demonstrated in Section 4 by numerical simulation and chaotic phenomena are observed. Finally, several concluding remarks are drawn in Section 5.

2. Linear stability analysis

In this section we consider system (3) with $c = 0$ described by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -kx(t) + af\big[x(t) - b(\tau)x(t-\tau)\big], \tag{4}$$

where $k > 0$, $a > 0$, $\tau \geqslant 0$ is the time delay and $b(\tau) > 0$, which is called memory function in this Letter, is a strictly decreasing function of τ.

Let $y(t) = x(t) - b(\tau)x(t-\tau)$, then Eq. (4) is recast into

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -ky(t) + af\big[y(t)\big] - ab(\tau)f\big[y(t-\tau)\big]. \tag{5}$$

If y_0 denotes an equilibrium of (5), then it satisfies that $ky_0 = [a - ab(\tau)]f(y_0)$.

Setting $u(t) = y(t) - y_0$, Eq. (5) can be written as

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -k\big(u(t) + y_0\big) + af\big[u(t) + y_0\big] - ab(\tau)f\big[u(t-\tau) + y_0\big]. \tag{6}$$

Throughout this Letter, we assume that the following conditions are satisfied:

$$f(0) = 0, \qquad f'(0) > 0, \quad f \in C^1(\mathrm{R}).$$

It is clear that Eq. (4) has the trivial equilibrium under the above condition.

The linear part of system (5) at its trivial equilibrium is

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -ku(t) + Au(t) - B(\tau)u(t-\tau), \tag{7}$$

where $A = af'(0)$, $B(\tau) = ab(\tau)f'(0)$. Since $f'(0) > 0$ is assumed to be true, then $A > 0$ and $B(\tau) > 0$ hold.

By rescaling time in Eq. (7) via $t = \tau\xi$ and setting $U(\xi) = u(\tau t)$, we obtain

$$\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi} = -k\tau U(\xi) + A\tau U(\xi) - \tau B(\tau)U(\xi - 1). \tag{8}$$

The corresponding characteristic equation of Eq. (8), obtained by substituting $U(\xi) = \exp(z\xi)$ into Eq. (8), is $H(z) = Pe^z + Q - ze^z = 0$, where $P = (A - k)\tau$ and $Q = -B(\tau)\tau$.

The stability of the trivial solution of Eq. (8) can be obtained by the result due to Hayes [22] taken from Bellman and Cook [23,24].

Lemma 1 ((Hayes)). *All the roots of $Pe^z + Q - ze^z = 0$, where P and Q are real, have negative real parts if and only if*: (1) $P < 1$, *and* (2) $P < -Q < \sqrt{a_1^2 + P^2}$, *where a_1 is the root of $a = P\tan(a)$ satisfying $0 < a < \pi$ (if $P = 0$, we take $a_1 = \pi/2$).*

In a mathematical sense, Lemma 1 gives us all the stability conditions we need on the parameters of the system. However, for some engineers and physicists who are mainly concerned with application, Lemma 1 is not immediately usable because, before the theorems can be applied, it is necessary to compute certain zeros of a transcendental equation. Furthermore, since we are interested in considering the effect of delay τ on the system dynamics, it is not easy to obtain the intervals of delay τ which guarantee the system (8) stability from Lemma 1, especially when the parameter b also depends on the delay τ. With an aim to make stability conditions more readily applicable, we will not make this time rescaling, but use the criteria of stability switches to analyze the stability of Eq. (4).

The characteristic equation, obtained by substituting $u(t) = \exp(\lambda t)$ into (7), is

$$D(\lambda) = \lambda + k - A + B(\tau)\exp(-\lambda\tau) = 0. \tag{9}$$

It is well known that the trivial equilibrium of system (7) is locally asymptotically stable if and only if each of the characteristic roots has negative real parts. Thus, the marginal stability is determined by the equations $\lambda = 0$ and $\lambda = \mathrm{i}\omega$ ($\omega > 0$). If