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Parametric resonance induced chaos in magnetic damped driven pendulum

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ABSTRACT

A damped driven pendulum with a magnetic driving force, appearing from a solenoid, where ac current flows is considered. The solenoid acts on the magnet, which is located at a free end of the pendulum. In this system the existence and interrelation of chaos and parametric resonance is theoretically examined. Derived analytical results are supported by numerical simulations and conducted experiments.

1. Introduction

Chaos in damped driven pendulum system has a long standing history (see e.g. Refs. [1,2] and references therein) and is applicable in vast variety of condensed matter problems [3–5] ranging from Josephson junctions to easy-plane ferromagnets. Governing equation is written in the standard form:

$$\ddot{\alpha} = -\Omega^2 \sin\alpha - q\dot{\alpha} + f_D \sin\omega t \tag{1}$$

where Ω and q coefficients are usually fixed and f_D is a one we control. Increasing control parameter f_D period doubling [6,7] bifurcation scenario and transition to chaos takes place [8–10]. In all the mentioned papers control parameter is constant [11] or a driving force has a time periodic singular character (kicked excited systems [12]). In the present paper driving force is position angle α dependent, particularly, here, a realistic example of driven damped pendulum model is considered. In this context, driving force is of a magnetic origin, particularly a solenoid with ac current is acting on the magnet, which plays a role of a bob in a pendulum with a rigid rod (see Fig. 1). Therefore the amplitude of a harmonic force f_D greatly depends on the distance between solenoid and the magnet, making it angle dependent in a non-trivial manner.

In the frames of the model (1) a possibility of onset of chaos has been examined analytically, numerically and experimentally. The similar model of magnetic pendulum has been studied long before [13], particularly, different orientation of solenoid and magnet has been considered, where the orientation of the solenoid is perpendicular to the pendulum's rod when the deviation angle is zero. In this case one gets quantization of amplitudes with no indication of onset of chaos, while in our case with parallel orientations of solenoid and pendulum in unperturbed position (see again Fig. 1) for some values of ac field and/or distance between solenoid and magnet chaos is observed due to the parametric resonance [14]. Thus the main peculiarity of our model is that the existence of parametric resonance is a necessary condition for the onset of chaos in the system.

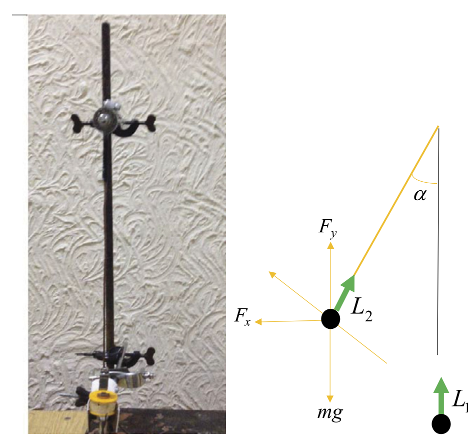

Fig. 1. Experimental setup (left) and schematics (right) for driven damped magnetic pendulum. L_1 is a dipolar moment of solenoid, L_2 is a dipolar moment of the magnet. α is a deviation angle of the pendulum from vertical. F_x and F_y are x and y components of magnetic force acting on the magnet.

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2. Theoretical model

In my experiments and numerical simulations the magnet is rigidly fixed in the place of a bob of the pendulum in such a way that the directions of its magnetic moment and the rod of pendulum coincide. Approximating solenoid and magnet as point-like magnetic moments ($\vec{L}_1$ and $\vec{L}_2$, respectively), one can readily write down their dipolar interaction energy as follows:

$$U = \frac{\mu_0}{4\pi}\left(\frac{3\cdot(\vec{L}_1\cdot\vec{r})(\vec{L}_2\cdot\vec{r})}{r^5} - \frac{\vec{L}_1\cdot\vec{L}_2}{r^3}\right)$$

where $\vec{r} \equiv (x,\ y)$ is a radius vector of magnet with respect to the solenoid, $r = \sqrt{x^2+y^2}$. Taking into account now that ac current is flowing into the solenoid and the magnet is attached at the free end of the pendulum one can write for the components of magnetic moments following expressions (see also Fig. 1):

$$L_{1x} = 0, \quad L_{2x} = -L_2\frac{x}{\ell},$$
$$L_{1y} = L_1(t), \quad L_{2y} = L_2\frac{r_0+\ell-y}{\ell} \tag{2}$$

where ℓ is the length of the pendulum and r_0 is distance between magnet and solenoid when the deviation angle from vertical direction is zero (that is a minimal distance position between solenoid and magnet).

Plugging (2) into (1) we find F_x and F_y components of the forces acting on the magnet:

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y}$$

We write Newton's second law for tangential axis of the pendulum as follows:

$$m\ddot{\alpha}\ell = F_x\cos\alpha + F_y\sin\alpha - mg\sin\alpha - q\dot{\alpha} \tag{3}$$

where a damping proportional to velocity has been included and m is a mass of the magnet. We do not write here explicit expressions for components of the force because of their cumbersomeness, although their complete expressions will be used for numerical simulations, while for analytics we just linearize (3) for small deviation angles α and approximate $r \to r_0$:

$$\ddot{\alpha} = -\alpha\left(\frac{g}{\ell} + \frac{12L_1(t)L_2}{mr_0^5} + \frac{2L_1(t)L_2}{m\ell^2 r_0^3}\right) - q\dot{\alpha} \tag{4}$$

where $L_1(t) \equiv L_1^0\cos 2\omega t$ because of the ac current (with 2ω frequency) flowing through the solenoid. Then let us denote

$$\omega_0 = \sqrt{\frac{g}{\ell}}, \qquad h = L_1^0\left(\frac{12L_2}{mr_0^5} + \frac{2L_2}{m\ell^2 r_0^3}\right) \tag{5}$$

and reduce (4) to the following equation:

$$\ddot{\alpha} = -\alpha(\omega_0^2 + h\cos 2\omega t) - q\dot{\alpha} \tag{6}$$

which is just a Mathieu equation if one sets damping to zero.

The presence of parametric resonance in (6) is examined in Ref. [15] for driving frequencies ω close to pendulum oscillation frequency ω_0. Actually, similar analysis could be done for arbitrary ω and the existence of parametric resonance in the system will cause undamped oscillations, chaos and some more interesting phenomena. In order to find out what conditions should be fulfilled for this to occur, we should seek the solution of equation (6) in the following form:

$$\alpha = a(t)\cos\omega t + b(t)\sin\omega t \tag{7}$$

Considering $a(t)$ and $b(t)$ as slow functions of time and neglecting their second derivatives, (6) is simplified to the following form:

$$X\cos\omega t + Y\sin\omega t = 0 \tag{8}$$

where coefficients X and Y both depend on $a(t)$ and $b(t)$. For the equation to be true, both coefficients should be equal to zero. Thus we get a set of two equations, where our goal is to find the regimes of parametric resonance. For this, we should seek for the solution in the exponential form $a(t) = Ae^{st}$ and $b(t) = Be^{st}$ and two equations are derived:

$$A\cdot(2s\omega + q\omega) - B\cdot({\omega_0}^2 + \frac{h}{2} - \omega^2) = 0$$
$$A\cdot(\omega^2 - \frac{h}{2} - {\omega_0}^2) - B\cdot(2s\omega + q\omega) = 0. \tag{9}$$

Finally we get from the compatibility condition:

$$s = \frac{{\omega_0}^2 + \frac{h}{2} - \omega^2 - q\omega}{2\omega} \tag{10}$$

Considering parametric instability growth rate s to be positive, the instability condition will be:

$$h \geq 2\mid w^2 - {w_0}^2 + 2q\omega\mid. \tag{11}$$

This defines the limits of existence of parametric resonance and its dependence on various parameters, but all of these is valid for small angles. In order to get the full dynamics we should solve differential equation (3) in a full range of angles. F_x and F_y components of magnetic force are known from derivative of dipole–dipole energy. If we do not consider the angle as small, we will not be able to make the approximations that has been done before. In general, F_x and F_y components are very complicated expressions and it is impossible to solve the equation (3) analytically. Therefore I performed numerical simulations using Matlab.

3. Numerical simulations

Our next goal is to prove theoretically the existence of chaos in the system, considering deviation angles as arbitrary. The given equation of motion (3) has been solved using *ode*45 toolbox of Matlab program with an initial guess that chaos should occur when parametric resonance for small angles takes place. And this appears to be true, because as the numerical simulations show, when there is parametric instability in the system, it is always chaotic. To prove the existence of chaos, the common way is to check, whether changing any parameter insignificantly, the difference between the first and second measurement of some variable increases exponentially in time. In other words, Lyapunov exponent should be calculated in order to analyze the behavior of chaotic motion. To calculate the exponent, one has to deviate e.g. initial angle $\alpha(0)$ by small value making it $\alpha'(0)$ and as time evolves, divide the resulting difference between angles $\alpha(t)$ and $\alpha'(t)$ on initial deviation. Taking out logarithm from this, dividing on time and averaging the results upon the initial deviations Lyapunov exponent of the process could be defined. Positive exponent is an obvious indication of the presence of chaos and one should look at the simultaneous presence of parametric resonance condition in the system.

Another test to check the relation between parametric resonance and chaos in our case of magnetic pendulum is to look whether the system performs large angle oscillations starting from initial insignificant deviations. In other words, if we give the pendulum very small initial angle, for example 0.0001 rad, and after some time it starts to oscillate with normal angles, this means that there is parametric resonance and chaos in the system. The