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Stochastic systems with delay: Perturbation theory for second order statistics

T.D. Frank *

Department of Psychology and Department of Physics, University of Connecticut, Storrs, CT 06269, USA

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ABSTRACT

Within the framework of delay Fokker–Planck equations, a perturbation theoretical method is developed to determine second-order statistical quantities such as autocorrelation functions for stochastic systems with delay. Two variants of the perturbation theoretical approach are presented. The first variant is based on a non-local Fokker–Planck operator. The second variant requires to solve a Fokker–Planck equation with source term. It is shown that the two variants yield consistent results. The perturbation theoretical approaches are applied to study negative autocorrelations that are induced by feedback delays and mediated by the strength of the fluctuating forces that act on the feedback systems.

1. Introduction

Time-delayed models have been used in various disciplines ranging from physics to the life sciences (for reviews see e.g. Refs. [1–6]). They are considered as approximative models that capture the main effect of a feedback loop but ignore details of mechanistic modeling. Time-delayed models have been studied in laser physics [7–13] and the engineering sciences [14–16]. The impact of time delayed feedback on bistable dynamical systems has been studied [17–19] and the possibility to control chaos [20] and noise-induced oscillations [21–23] by means of time-delayed feedback has been discussed. Time-delayed models have been used to address various problems in biology [3], such as motor control under delayed feedback [24–33], neural network systems under the impact of delay lines [34–41], the role of delays in ecological systems [1,42,43], and cell signaling and gene regulation with delays [44–47]. In addition, anticipation in master–slave systems involving coupling delays has been studied recently [48–51]. The stochastic case of time-delayed systems is of particular interest in physics and the life sciences because the systems of interests are frequently subjected to thermal noise fluctuations or random perturbations originating from other sources [6,52]. Research on stochastic delay systems has produced a substantial amount of interesting results (see references above), in particular, using the extended phase space approach [10,14,53–55] and the so-called delay Fokker–Planck equation method [4,56–61]. In the context of the latter approach, small time delay approximation methods have been proposed [4,56,62] and a perturbation theoretical technique has been developed to deal with time-delayed feedback loops that weakly impact a Markovian dynamics described by a Langevin equation [59,63]. Since then these approximative solution techniques based on delay Fokker–Planck equations have been applied in various studies [64–72].

However, the aforementioned perturbation theoretical method [59,63] has been developed to yield first order statistical quantities such as mean values and single time point probability densities. Therefore, at issue is to generalize the approach to address second order statistical quantities and in particular to derive autocorrelation functions in the time domain. In Sec. 2.1 we will outline a general perturbation theoretical method to derive second order statistical quantities within the framework of delay Fokker–Planck equations. In Sections 2.2 and 2.3, we will then focus on the stationary case and on very long time delays. In Sec. 2.4 we will show that the method yields results consistent with the exact solutions known for linear time-delayed models. An application to nonlinear stochastic time-delayed systems will be discussed in Sec. 2.5 and in this context, in Sections 2.6 and 2.7 the perturbation theoretical method will be used to discuss the emergence of negative autocorrelations induced by time-delayed feedback loops and mediated by the strength of fluctuating forces.

* Fax: +1 860 486 2760.
E-mail address: till.frank@uconn.edu.

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2. Perturbation theory for stochastic delay differential equations

2.1. General case

Let $X(t)$ denote a time-dependent random variable $X(t)$ defined on a domain Ω, where t is time. Let us assume that $X(t)$ satisfies the stochastic delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = h^{(0)}(X(t)) + \beta h^{(1)}(X(t), X(t-\tau)) + g(x)\Gamma(t), \tag{1}$$

where $h^{(0)}(x)$ is the unperturbed drift function, while the expression $\beta h^{(1)}(x, y)$ describes a perturbation imposed on the unperturbed dynamics. Here and in what follows $\beta \geq 0$ is a small (perturbation) parameter. The unperturbed dynamics corresponds to a Markov diffusion process given in terms of a Langevin equation. Accordingly, the term $g(x)\Gamma(t)$ is the fluctuating force acting on the evolution of X. $\Gamma(t)$ is a Langevin force [73,74] normalized to 1 like $\langle\Gamma(t)\Gamma(t')\rangle = \delta(t-t)$. Here and in what follows, $\delta(\cdot)$ is the Dirac delta function and $\langle\cdot\rangle$ means ensemble averaging. The function $g(x)$ is a measure for the strength of the fluctuating force. As indicated, we consider the case in which g can depend on the current state $X(t)$. The multiplicative fluctuating force $g(x)\Gamma(t)$ is interpreted using the Ito rule of stochastic differential equations [73]. The perturbation function $h^{(1)}$ is assumed to depend in general both on the current state $X(t)$ and on a previous state $X(t-\tau)$ of the system, where $\tau \geq 0$ denotes the time delay. Consequently, $\beta h^{(1)}(x, y)$ describes a delayed feedback loop that has a weak impact on the unperturbed dynamics.

The single variable probability density $P(x, t)$ is formally defined by $P(x, t) = \langle\delta(x - X(t))\rangle$. Likewise, the joint probability densities

$$\begin{aligned} &P(x, t; x', t') = \left\langle\delta(x - X(t))\delta(x' - X(t'))\right\rangle, \\ &P(x, t; x', t'; y, t-\tau) = \\ &\quad \left\langle\delta(x - X(t))\delta(x' - X(t'))\delta(y - X(t-\tau))\right\rangle \end{aligned} \tag{2}$$

with $t \geq t'$ can be introduced. The evolution equation for $P(x, t; x', t')$ reads (see Eq. (19) in Ref. [59])

$$\frac{\partial}{\partial t}P(x, t; x', t') = \int_\Omega \mathrm{d}y\, \hat{L}(x, y, \nabla_x)P(x, t; x', t'; y, t-\tau) \tag{3}$$

with the Fokker–Planck-like operator

$$\hat{L}(x, y, \nabla_x) = -\frac{\partial}{\partial x}[h^{(0)}(x) + \beta h^{(1)}(x, y)] + \frac{\partial^2}{\partial x^2}\frac{g^2(x)}{2}. \tag{4}$$

Equation (3) has been introduced by Guillouzic et al. [4,56,57] and will be referred to as delay Fokker–Planck equation [58,59,61]. It can be broken down into two parts

$$\begin{aligned} \frac{\partial}{\partial t}P(x, t; x', t') &= \hat{F}^{(0)}(x, \nabla_x)P(x, t; x', t') \\ &\quad - \frac{\partial}{\partial x}\int_\Omega \mathrm{d}y\, \beta h^{(1)}(x, y)P(x, t; x', t'; y, t-\tau) \end{aligned} \tag{5}$$

with the Fokker–Planck operator of the unperturbed system

$$\hat{F}^{(0)}(x, \nabla_x) = -\frac{\partial}{\partial x}h^{(0)}(x) + \frac{\partial^2}{\partial x^2}\frac{g^2(x)}{2}. \tag{6}$$

The objective is to derive a evolution equation for $P(x, t; x', t')$ that formally has the structure of a Fokker–Planck equation. To this end, we decompose the 3-time point joint probability density $P(x, t; x', t'; y, t-\tau)$ such that we can extract the function $P(x, t; x', t')$:

$$P(x, t; x', t'; y, t-\tau) = P(y, t-\tau|x, t; x', t')P(x, t; x', t'). \tag{7}$$

Here $P(y, t-\tau|x, t; x', t')$ is a conditional probability density defined by

$$P(y, t-\tau|x, t; x', t') = \langle\delta(x - X(t-\tau))\rangle|_{X(t)=x, X(t')=x'}, \tag{8}$$

where $\langle\cdot\rangle|_G$ is an ensemble average under the constraint defined by the condition G. Note that Eq. (7) formally satisfies the well-known relation $P(A, B, C) = P(C|A, B)P(A, B)$ that relates any three variable joint probability function $P(A, B, C)$ to a corresponding two variable joint probability function $P(A, B)$ by means of the condition probability density $P(C|A, B)$. Substituting Eq. (7) into Eq. (5), we find

$$\begin{aligned} &\frac{\partial}{\partial t}P(x, t; x', t') = \\ &\quad \left\{\hat{F}^{(0)}(x, \nabla_x) - \frac{\partial}{\partial x}\int_\Omega \mathrm{d}y\, \beta h^{(1)}(x, y)P(y, t-\tau|x, t; x', t')\right\} \\ &\quad \times P(x, t; x', t'), \end{aligned} \tag{9}$$

which is the desired Fokker–Planck-like evolution equation for $P(x, t; x', t')$.

In what follows, we will proceed in analogy to the perturbation theoretical method for the first order statistics developed earlier [59]. Accordingly, we put

$$\begin{aligned} P(x, t; x', t') &= P^{(0)}(x, t; x', t') + \beta\epsilon(x, t; x', t') \\ &\quad + \beta^2\chi(x, t; x', t') + O(\beta^2). \end{aligned} \tag{10}$$

$P^{(0)}(x, t; x', t')$ is the joint probability density in zeroth order of the small parameter β. The function ϵ is the first order correction function. In Eq. (10), the function χ is a second order correction term but it does not capture all second order effects (see below and Ref. [59]). The remaining second order and higher order terms are captured by the expression "$O(\beta^2)$". Let us introduce $P^{(0)}(y, t-\tau|x, t; x', t')$ as the zeroth order approximation of the conditional probability density $P(y, t-\tau|x, t; x', t')$. Furthermore, let us introduce $\zeta(y, x, x')$ as the corresponding first order correction term of $P(y, t-\tau|x, t; x', t')$ such that $P(y, t-\tau|x, t; x', t') = P^{(0)}(y, t-\tau|x, t; x', t') + \beta\zeta(y, x, x') + O(\beta^2)$ holds. Substituting Eq. (10) and the expansion for $P(y, t-\tau|x, t; x', t')$ into Eq. (9) yields

$$\begin{aligned} &\frac{\partial}{\partial t}P^{(0)}_{xx'} + \beta\frac{\partial}{\partial t}\epsilon + \beta^2\frac{\partial}{\partial t}\chi + O(\beta^2) = \hat{F}^{(0)}\left(P^{(0)}_{xx'} + \beta\epsilon + \beta^2\chi\right) \\ &\quad - \frac{\partial}{\partial x}\int_\Omega \mathrm{d}y\, \beta h^{(1)}\left[P^{(0)}_{yxx'} + \beta\zeta\right]\left(P^{(0)}_{xx'} + \beta\epsilon + \beta^2\chi\right) + O(\beta^2), \end{aligned} \tag{11}$$

where we used the short notations $P^{(0)}_{xx'}$ and $P^{(0)}_{yxx'}$ for $P^{(0)}_{xx'} = P^{(0)}(x, t; x', t')$ and $P^{(0)}_{yxx'} = P^{(0)}(y, t-\tau|x, t; x', t')$. Collecting all terms of zeroth order in β, we see that the joint probability density in zeroth order, $P^{(0)}(x, t; x', t')$, satisfies

$$\frac{\partial}{\partial t}P^{(0)}(x, t; x', t') = \hat{F}^{(0)}P^{(0)}(x, t; x', t'). \tag{12}$$

Collecting all terms linear in β, we see that the first order correction term, $\epsilon(x, t; x', t')$, can be computed from

$$\begin{aligned} \frac{\partial}{\partial t}\epsilon(x, t; x', t') &= \hat{F}^{(0)}(x, \nabla_x)\epsilon(x, t; x', t') \\ &\quad - \frac{\partial}{\partial x}\int_\Omega \mathrm{d}y\, h^{(1)}(x, y)P^{(0)}(y, t-\tau|x, t; x', t')P^{(0)}(x, t; x', t'). \end{aligned} \tag{13}$$