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# Electrodynamics of moving magnetoelectric media: Variational approach

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**Abstract**

Recently, Feigel has predicted a new effect in magnetoelectric media. The theoretical evaluation of this effect requires a careful analysis of a dynamics of the moving magnetoelectric medium and, in particular, the derivation of the energy–momentum of the electromagnetic field in such a medium. Then, one can proceed with the study of the wave propagation in this medium and derive the mechanical quantities such as the energy, the momentum, and their fluxes and the corresponding forces. In this Letter, we develop a consistent general-relativistic variational approach to the moving dielectric and magnetic medium with and without magnetoelectric properties. The old experiments in which the light pressure was measured in fluids are reanalysed in our new framework.


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## 1. Introduction

The discussion of the electrodynamics of moving media has a long history. At present, the general structure of classical electrodynamics appears to be well established. In particular, in the generally covariant pre-metric approach to electrodynamics [1–6], the electric charge and the magnetic flux conservation laws manifest themselves in the Maxwell equations for the excitation $H = (\mathcal{D}, \mathcal{H})$ and the field strength $F = (E, B)$, namely $dH = J$, $dF = 0$. These equations should be supplemented by a constitutive law $H = H(F)$. The latter relation contains the crucial information about the underlying physical continuum (i.e., spacetime and/or material medium), in particular, about the spacetime metric. Mathematically, this constitutive law arises either from a suitable phenomenological theory of a medium or from the electromagnetic field Lagrangian. It can be a nonlinear or even nonlocal relation between the electromagnetic excitation and the field strength. The constitutive law is called a spacetime relation if it applies to spacetime ("the vacuum") itself.

Among many physical applications of classical electrodynamics, the problem of the interaction of the electromagnetic field with matter occupies a central position. The fundamental question, which arises in this context, is about the definition of the energy and momentum in the possibly moving medium. The discussion of the energy–momentum tensor in macroscopic electrodynamics is quite old. The beginning of this dispute dates back to Minkowski [7], Einstein and Laub [8], and Abraham [9]. Nevertheless, up to now the question was not settled and there is an on-going exchange of conflicting opinions concerning the validity of the Minkowski versus the Abraham energy–momentum tensor, see, e.g., the review [10]. Even experiments were not quite able to make a definite

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and decisive choice of electromagnetic energy and momentum in material media. A consistent solution of this problem has been proposed in [4,11] (cf. also the earlier work [12]) in the context of a new axiomatic approach to electrodynamics.

Recently Feigel [13] has studied, theoretically, the dynamics of a dielectric magnetoelectric medium in an external electromagnetic field. He predicted that the contributions of the quantum vacuum waves (or "virtual photons") could transfer a nontrivial momentum to matter. This prediction was made with the help of the non-relativistic formalism. In our opinion, a proper relativistic analysis is needed for a better understanding of the physics and of the viability of this phenomenon. Here we begin to reconsider this problem in a covariant framework as developed earlier in [4,11]. As a first step, we develop a variational approach to the description of the dynamics of a moving magnetoelectric medium. The corresponding energy–momentum of matter plus electromagnetic field that arises can be derived straightforwardly in this formalism from the variation of the total action with respect to the spacetime metric.

## 2. Preliminaries: The essence of the Feigel effect

The Feigel effect [13] can be described in simple terms as follows: Let us consider an isotropic homogeneous medium with the electric and magnetic constants $\varepsilon, \mu$. Electromagnetic waves are propagating in such a medium absolutely symmetrically, with the Fresnel equation describing the unique light cone. This is easily derived from the constitutive relations $\mathcal{D} = \varepsilon\varepsilon_0 E$ and $\mathcal{H} = (\mu\mu_0)^{-1} B$.

However, if a medium is placed in crossed constant external electric and magnetic fields, then it acquires magnetoelectric properties. As a result, we have the *anisotropic* magnetoelectric medium with $\varepsilon$, $\mu$, plus the magnetoelectric matrix $\beta$ (determined by the external fields) which modifies the constitutive relations to $\mathcal{D} = \varepsilon\varepsilon_0 E + \beta \cdot B$ and $\mathcal{H} = (\mu\mu_0)^{-1} B - \beta^{\mathrm{T}} \cdot E$; here $^{\mathrm{T}}$ denotes the transposed matrix.

Accordingly, the wave propagation in such a medium also becomes anisotropic and birefringent, with the wave covectors now belonging to two light cones. Applying this to vacuum waves (or, perhaps, better to say to the "vacuum fluctuations" or "virtual photons") propagating in the magnetoelectric body, Feigel [13] computed the total momentum carried by these waves and concluded that it is nontrivial. In accordance with this derivation, a body should move with a small but non-negligible velocity. Earlier the Feigel process was discussed in [14–18].

In order to evaluate the possible Feigel effect, it is necessary to substitute the "vacuum waves" into the energy–momentum tensor. This Letter is devoted to the derivation of the latter in the framework of a variational approach.

## 3. Constitutive relation

Within the axiomatics of the pre-metric generally covariant framework [4], the projection technique is used to define the electric and magnetic phenomena in an arbitrarily *moving* medium. As in [4], we assume that the spacetime is foliated into spatial slices with time $\sigma$ and transverse vector field $n$.

When applying the projection technique to the 2-forms of the electromagnetic excitation $H$ and the electromagnetic field strength $F$, we obtain the three-dimensional objects: the magnetic and electric excitations $\mathcal{H}$ and $\mathcal{D}$ as longitudinal and transversal parts of $H$ and, similarly, electric and magnetic fields $E$ and $B$ as longitudinal and transversal parts of $F$, respectively, namely

$$H = -\mathcal{H} \wedge d\sigma + \mathcal{D} \quad \text{and} \quad F = E \wedge d\sigma + B. \tag{3.1}$$

This foliation is called the *laboratory* foliation, with the coordinate time variable $\sigma$ labeling the slices of this foliation.

The spacetime metric $\mathbf{g}$ introduces the scalar product in the tangent space and defines the line element. With respect to the laboratory foliation coframe it reads $(a, b, \ldots = 1, 2, 3)$

$$ds^2 = N^2\, d\sigma^2 + g_{ab}\, \underline{dx}^a\, \underline{dx}^b = N^2\, d\sigma^2 - {}^{(3)}g_{ab}\, \underline{dx}^a\, \underline{dx}^b. \tag{3.2}$$

Here $N^2 = \mathbf{g}(n, n)$ is the length square of the foliation vector field $n$, and $\underline{dx}^a = dx^a - n^a\, d\sigma$ is the transversal 3-covector basis, in accordance with the definitions above. The 3-metric ${}^{(3)}g_{ab}$ is the positive definite Riemannian metric on the spatial 3-dimensional slices corresponding to fixed values of the time $\sigma$. This metric defines the 3-dimensional Hodge duality operator $\underline{\star}$.

The constitutive relation which links the electromagnetic field strength to the electromagnetic excitation, $H = H(F)$, can be nonlocal and nonlinear, in general. Here we will confine our attention to the local and linear constitutive relation.

Then, if we write the excitation 2-form in terms of its components in a local coordinate system $\{x^i\}$, $(\mathcal{H}, \mathcal{D}) = H = H_{ij}\, dx^i \wedge dx^j/2$ (with $i, j, \ldots = 0, 1, 2, 3$), the local and linear constitutive relation means that the components of the excitation are local linear functions of the components of the field strength $(E, B) = F = F_{ij}\, dx^i \wedge dx^j/2$:

$$H = \kappa(F), \qquad H_{ij} = \frac{1}{2}\kappa_{ij}{}^{kl} F_{kl}. \tag{3.3}$$

Along with the original constitutive $\kappa$-tensor, it is convenient to introduce an alternative representation of the constitutive tensor:

$$\chi^{ijkl} := \frac{1}{2}\epsilon^{ijmn}\kappa_{mn}{}^{kl}. \tag{3.4}$$