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Complete multiplier statistics explained by stochastic cascade processes

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Abstract

The prevalent understanding of fully developed turbulence is that of a cascade in which vortices successively break up into smaller ones. Based on this idea, theoretical developments were often concerned with multiplicative random cascade processes. We focus on the velocity fluctuations and derive a closed as well as complete statistical description of the underlying velocity multipliers using a Fokker–Planck equation, which is estimated directly from experimental data. This shed new light on the statistics of multipliers and their often assumed independence. For the heavy-tailed statistical features of the multipliers, close to a Cauchy distribution, no intermittency of turbulence is needed.

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Intermittency, i.e. the extraordinary frequent occurrence of strong velocity fluctuations on small scales, is the most debated problem in fully developed turbulence. The predominant understanding of this phenomenon is based on a cascading process, where vortices successively break up into smaller ones, although Richardson's original cascade picture has undergone substantial changes during the last time, cf. [1,2]. The description of this transition from large to small scales by an iterative multiplication with random variables—a multiplicative random cascade process—had led to many theoretical developments [1,3,4]. Similar processes also describe complex hierarchical systems encountered in other fields of science such as economy [5], geology [6] and computer science [7].

In turbulence multiplicative processes describe either the energy cascade or the velocity fluctuations [1,8–10]. In this Letter we focus on velocity multipliers

$$w_n := \frac{u_{n+1}}{u_n} \tag{1}$$

as the ratio of longitudinal velocity increments $u_n \equiv u(x + r_n/2) - u(x - r_n/2)$ over a scale r_n.

Kolmogorov has formulated in his famous 1962s paper three alternative hypotheses to the initially provided ones in the log-normal model which are based on assumptions of velocity multipliers [10]. They state in essential that multipliers are statistically independent for large scale-separation (third hypothesis) and that they are scale-invariant for high Reynolds numbers. But his predicted log-normality cannot be found in practice. In [9] it was pointed out that the multipliers' distribution of velocity increments is Cauchy-distributed and has short correlations.

The statistical properties of multipliers play a fundamental role in the scaling theory of multiscale correlation functions and the therein used fusion rules. A central assumption in this theory is that the velocity fluctuations on small scales are statistically independent of the velocity fluctuations on large scales or equivalent that the multipliers are uncorrelated. Corrections to the predicted scaling behavior are given by geometric constrains ('ward identities'), see for example [11,12].

In this Letter we estimate the complete multiplier statistics from experimental data and give an explicit expression of the n-scale joint probability distribution $p(w_n, \dots, w_1)$. This allow us to discuss in detail the important role of correlations and the

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contribution of intermittency. We derive the multiplier statistics by using the observation that the statistics of the small-scale turbulent velocity field can be described by a Fokker–Planck equation [13–15].

We verify our results with two experimental data sets. The first one is measured in a wake behind a cylinder. The Reynolds number is 13 000 and the set consists of 1.25×10^6 data points; for further details see [16]. The results presented here are from this data set. For comparison we use data measured in a cryogenic free jet at Reynolds number 757 000, for further details see [14,17].

1. Fokker–Planck description of the turbulent cascade

The small-scale turbulence can be characterized by velocity increments $u_r := u(x + \frac{r}{2}) - u(x - \frac{r}{2})$, which describe the fluctuations of the velocity field on different scales r. Increment statistics are often characterized by means of structure functions $\langle u_r^n \rangle \propto r^{\xi_n}$, which are mainly equivalent to the simple probability distribution $p(u_r, r)$. For a more detailed examination of the small-scale turbulence it is of interest to consider joint probabilities, i.e. $p(u_n, r_n; u_{n-1}, r_{n-1}; \ldots; u_1, r_1)$, where u_i are the increments on the scale r_i with common reference point x. Due to the large number of variables involved, this joint distribution cannot be estimated directly from data. But in [13] and references therein, it has been shown that the increments u_r obey a Markov process in r, i.e. the joint probability can be expressed by the conditional probabilities,

$$\begin{aligned} &p(u_n, r_n; u_{n-1}, r_{n-1}; \ldots; u_1, r_1) \\ &\quad = p(u_n, r_n | u_{n-1}, r_{n-1}) \cdots p(u_2, r_2 | u_1, r_1) p(u_1, r_1). \end{aligned} \tag{2}$$

The conditional distributions $p(u_{n+1}, r_{n+1} | u_n, r_n)$ are given by a Kramers–Moyal expansion, which, as it was shown for example in [13], truncates after the second term, resulting in a Fokker–Planck equation[1]

$$\begin{aligned} &-r \frac{\partial p(u, r | u', r')}{\partial r} \\ &\quad = \left(-\frac{\partial}{\partial u} D^{(1)}(u, r) + \frac{\partial^2}{\partial u^2} D^{(2)}(u, r) \right) p(u, r | u', r'). \end{aligned} \tag{3}$$

Both coefficients, the drift term $D^{(1)}$ and the diffusion term $D^{(2)}$, can be estimated via the Kramers–Moyal coefficients directly from measured time series, see [16]. Typically, the coefficients have the following dependence on u: $D^{(1)}(u, r) = \gamma(r) u$, $D^{(2)}(u, r) = \alpha(r) + \alpha^u(r) u + \alpha^{uu}(r) u^2$ (the upper u is an index denoting the order of the coefficient). The term $\alpha^u(r) u$ describes the skewness of the probability distribution, the term $\alpha^{uu}(r) u^2$ the intermittency. The r-dependence is given by $\alpha(r) = \alpha_1 r$ and $\gamma(r) = \gamma_0 + \gamma_1 r$, see [16].

[1] Note that, in contrast to the usual definition as, for example, given by [21], we multiply both sides of the Kramers–Moyal expansion by the scale r. This is equivalent to the logarithmic length scale $\lambda = \ln(L/r)$ used by [15]. The negative sign of the left-hand side is due to the direction of the cascade toward smaller scales r.

It is important to note that every step of the derivation of the Fokker–Planck equation has been checked by us on data. This means that there are no assumptions in the derivation which are not compatible to experiments. In detail: we have tested the Markovian properties for different scales and for different flows, they are well fulfilled. Higher order Kramers–Moyal coefficients are small so that the general expansion reduces to the Fokker–Planck equation. Finally, it has been shown that the increments' distribution, as well as the joint distribution of the measured increments is well described by the Fokker–Planck equation.

Next, we consider an essential simplification and neglect the skewness and the intermittency term of the diffusion coefficient, i.e. we set $D^{(1)}(u, r) = \gamma(r) u$ and $D^{(2)}(u, r) = \alpha(r)$. This is motivated by the fact that the Cauchy distribution results from the ratio of symmetric, normal distributed stochastic variables; a detailed explanation is given below. Then, the solution of the Fokker–Planck equation can be given by a Gaussian distribution

$$p(u, r | u', r') = N \exp\left(-\frac{u^2}{a} + \frac{2bu'}{a} u \right). \tag{4}$$

Inserting this in the Fokker–Planck equation (3), the functions a, b and N are given by the differential equations

$$r \frac{\partial N}{\partial r} = -\gamma N + 4 \frac{\alpha b^2 u'^2}{a^2} N + 2 \frac{\alpha}{a} N, \tag{5}$$

$$r \frac{\partial b}{\partial r} = \gamma b \tag{6}$$

and

$$r \frac{\partial a}{\partial r} = 2\gamma a + 4\alpha. \tag{7}$$

The solutions of these differential equations are:

$$a(r) = -4\alpha_1 \int_{r'}^{r} \left(\frac{r}{s} \right)^{-2\gamma_0} \exp\bigl(-2\gamma_1 (r - s) \bigr) \, ds, \tag{8}$$

$$b(r) = \left(\frac{r}{r'} \right)^{-\gamma_0} \exp\bigl(-\gamma_1 (r - r') \bigr). \tag{9}$$

The equation for N is just the normalization condition

$$N = \frac{1}{\sqrt{a\pi}} \exp\bigl(-b^2 u'^2 / a \bigr). \tag{10}$$

So far we have used the Markovian properties of the velocity increments and their statistical description by a Fokker–Planck equation to derive an expression for the conditional probability distribution $p(u, r | u', r')$, see Eq. (4). Next, we will use this result to derive the distribution of the multipliers and their multi-scale statistics.

2. Derivation of the multipliers' Cauchy distribution

From the Fokker–Planck equation the statistics for each combination of increments can be derived. The distribution $p(w)$ of the multiplier w_n, see Eq. (1), can be derived from