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Ergodicity in a two-dimensional self-gravitating many-body system

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ABSTRACT

We study the ergodic properties of a two-dimensional self-gravitating system using molecular dynamics simulations. We apply three different tests for ergodicity: a direct method comparing the time average of a particle momentum and position to the respective ensemble average, sojourn times statistics and the dynamical functional method. For comparison purposes they are also applied to a short-range interacting system and to the Hamiltonian mean-field model. Our results show that a two-dimensional self-gravitating system takes a very long time to establish ergodicity. If a Kac factor is used in the potential energy, such that the total energy is extensive, then this time is independent of particle number, and diverges with $\sqrt{N}$ without a Kac factor.

1. Introduction

N-body systems with long-range interactions have some peculiarities with respect to systems with short-range interactions, having drawn much attention at least along the last two decades [1–5], and for a longer time if one considers self-gravitating systems and charged plasmas. Starting from a non-equilibrium configuration, a short-range interacting system evolves to thermodynamic equilibrium in a relatively small relaxation time, while a long-range interacting system evolves over different stages characterized by time-scales differing by orders of magnitude, and taking a very long time to reach thermodynamic equilibrium for a finite number of particles. The initial stage of evolution is a violent relaxation into a Quasi-Stationary State (QSS) in a time roughly independent of the number of particles [6]. The relaxation time to thermodynamic equilibrium diverges in the $N \to \infty$ limit (Vlasov limit) and gets trapped in a non-equilibrium non-Gaussian stationary state.

A pair interaction potential is long-ranged if it decays at long distances as $r^{-\gamma}$ with $\gamma < d$, with d the spatial dimension and r the inter-particle distance. This implies that all particles, no matter how far, contribute to the total energy. Consequently the system is non-additive, violating the fundamental axiom of thermodynamics: non-additivity of entropy. It is worth remembering that this is not in contradiction with the second law of thermodynamics, which is always valid [7]. Another common consequence is non-ergodicity, most extensively studied for the Hamiltonian mean field model [8, 11–17], but also for one- and three-dimensional self-gravitating systems [19–23]. Quite curiously, and up to the authors knowledge, no previous detailed study was devoted to the ergodicity of the two-dimensional case, at least not in the sense discussed here. In the Vlasov limit, the dynamics of the system is exactly described by the Vlasov equation, which is essentially the Liouville equation for the one-particle distribution function evolving in the mean-field due to all other particles, and given by [24,25]:

$$\frac{d}{dt} f(\mathbf{p}, \mathbf{r}; t) = \left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} + \mathbf{F}(\mathbf{r}; t) \cdot \frac{\partial}{\partial \mathbf{p}} \right) = 0, \tag{1}$$

where $f(\mathbf{p}, \mathbf{r}; t)$ is the one-particle distribution function, $\mathbf{p}$ and $\mathbf{r}$ the momentum and position vectors in d spatial dimensions, respectively, and

$$\mathbf{F}(\mathbf{r}; t) = -\frac{\partial}{\partial \mathbf{r}} \int d\mathbf{r}' \, V(\mathbf{r} - \mathbf{r}'), \tag{2}$$

is the mean field force at position $\mathbf{r}$ and time t with $V(\mathbf{r} - \mathbf{r}')$ the inter-particle potential. For finite N the right-hand side of Eq. (1) does no longer vanish and is given by the pertinent collisional contributions (granularity effects) [26]. For a long-range interacting system the thermodynamic limit is ill defined [27], and the Vlasov limit can be taken consistently by introducing a Kac factor $1/N$ in the interaction potential, such that the energy becomes extensive, but still non-additive, and that the $N \to \infty$ limit converges [28].

The study of ergodicity was pioneered by Boltzmann in his works on the foundations on Statistical Mechanics [29], and latter extended by Birkhoff [30] and Khinchin [31]. The ergodic hypothesis states that time average of a dynamical functions $b(x)$ equals the ensemble average:

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$$\lim_{t_f\to\infty}\frac{1}{t_f-t_0}\int_{t_0}^{t_f} \mathrm{d}t\, b(x_t)\equiv \overline{b}(x)=\langle b(x)\rangle\equiv\int \mathrm{d}\mu_0\, b(x), \tag{3}$$

where x denotes a point in the system state space (or phase space for a Hamiltonian classical system), $\overline{b}(x)$ is the time average of $b(x)$, $\langle b(x)\rangle$ its ensemble average and $d\mu_0$ a statistical measure. Proving ergodicity rigorously is a difficult task and has been accomplished only for a few cases. Most studies rely on different methods such as determining the existence of gaps in phase-space [11,12], direct comparison of time and ensemble averages for the momentum variable [13], sojourn time statistics for cells in phase-space [18,32,33], testing for equipartition of energy [19,20,34] and the dynamical functional approach [35–37]. The latter can also be used to determined weather the system is mixing, which is a stronger property than ergodicity. The dynamics of a system is mixing in its phase-space S if for an invariant measure μ, $\mathbf{T}$ a map preserving μ and $s_1, s_2 \subset S$, we have that [39]:

$$\frac{\mu(s_1)}{\mu(S)}=\lim_{k\to\infty}\frac{\mu(s_2\cap \mathbf{T}^k s_1)}{\mu(s_2)}. \tag{4}$$

Non-ergodicity can also be classified as strong if some regions of phase-space are non-accessible to the system and weak if all regions are accessible but not equally visited.

Ergodicity can be studied in two complementary ways. The first possibility is to consider the evolution in phase space of the N-particle system, with time averages computed over the trajectory of the point representing the state of the whole system. Ensemble averages are then obtained by considering all points in the energy hypersurface as equally probable, i.e. for the microcanonical equilibrium. This is equivalent to state that the total time spent in a phase-space region is asymptotically proportional to its measure. The other possibility is to consider single particle histories as realizations of a stochastic process. In this case time averages are taken over one particle history, while ensemble averages are taken over the set of all N particles. This approach is particularly suited for a direct experimental verification even for short-range interacting systems [38]. In this case ergodicity, i.e. the coincidence of the time and ensemble averages is not equivalent to thermodynamic equilibrium as will become evident below. We consider the latter case in the present work, with S in Eq. (4) the one-particle phase-space, for a system composed by a single particle evolving in a stationary potential, either in the thermodynamic equilibrium or in a (quasi-)stationary state.

In this paper we first briefly study a gas of elastic hard-discs as an example of short-range interacting system, and then revisit the ergodic properties of the Hamiltonian Mean Field (HMF) model extending the results of Ref. [13] to include non-homogeneous quasi-stationary states. Both system are used as comparison standards for our main interest here, a two-dimensional self-gravitating N-body system. The HMF model is formed by N particles on a circle with Hamiltonian [8]:

$$H=\sum_{k=1}^{N}\frac{p_k^2}{2}+\frac{1}{2N}\sum_{k,l=1}^{N}\left[1-\cos\left(\theta_k-\theta_l\right)\right], \tag{5}$$

where θ_k is the position angle on particle k and p_k its conjugate momentum. It is a solvable model at equilibrium [8,9] at its dynamics can be simulated with a smaller computational cost if compared to other long-range interacting systems [10] and widely studied in the literature (see [1–4] and references therein). A two-dimensional self-gravitating system with N identical particles, as described by the Hamiltonian:

$$H=\sum_{k=1}^{N}\frac{p_k^2}{2}+\frac{1}{2N}\sum_{k,l=1}^{N}\ln\left(|\mathbf{r}_k-\mathbf{r}_l|+\epsilon\right), \tag{6}$$

where the logarithmic potential is the solution of the Poisson equation in two dimensions [42], all masses are set to unity and ϵ is a softening small parameter, commonly used in simulations of self-gravitating systems to avoid the divergence in the potential at zero distance [55]. Note that we use a Kac factor $1/N$ in the potential.

The potential in Eq. (6) is not upper bound and is therefore confining, consequently avoiding the difficulty of particle evaporation in three-dimensional gravity. This model was used for instance in the study of anomalous diffusion in a collapsing phase [43], the determination of thermodynamic equilibrium properties [44], collisional relaxation [25,45], violent relaxation [42,46] and cooling in self-gravitating accretion discs [47].

The structure of the paper is as follows. In Section 2 we succinctly present the methods for testing ergodicity used in the present work, and apply then to a two-dimensional hard-disc gas in Section 3, and to the HMF model in Section 4. The ergodic properties of a two-dimensional self-gravitating system are discussed in Section 5. We close the paper with some concluding remarks in Section 6.

2. Testing ergodicity

2.1. Direct method

We consider the time evolution of a single particle to compute time averages while ensemble averages are obtained by taking the average over the N particles in the whole system. For an ergodic system both quantities coincide. Let us then consider the momentum $p_k(t)$ of particle k and its position $x_k(t)$ at time t and write their time averages as:

$$\overline{p}_k(t)=\frac{1}{n}\sum_{j=0}^{n}p_k(j\Delta t), \tag{7}$$

and

$$\overline{x}_k(t)=\frac{1}{n}\sum_{j=0}^{n}x_k(j\Delta t), \tag{8}$$

respectively, where Δt is a constant time interval, that we take as being the numeric integration time step and m is the total number of such intervals. The time averages are computed up to a given time t denoted by the argument of $\overline{p}_k(t)$ and $\overline{x}_k(t)$. On the other hand, ensemble averages at time t are given by

$$\langle p(t)\rangle=\frac{1}{N}\sum_{i=0}^{N}p_i(t), \tag{9}$$

and

$$\langle x(t)\rangle=\frac{1}{N}\sum_{i=0}^{N}x_i(t). \tag{10}$$

For an ergodic system $\overline{p}_k$ and $\overline{x}_k$ are the same for all particles, and therefore the standard deviations:

$$\sigma_{\overline{p}}(t)=\sqrt{\frac{1}{N}\sum_{k=1}^{N}\overline{p}_k(t)^2-\langle\overline{p}(t)\rangle^2}, \tag{11}$$

and

$$\sigma_{\overline{x}}(t)=\sqrt{\frac{1}{N}\sum_{k=1}^{N}\overline{x}_k(t)^2-\langle\overline{x}(t)\rangle^2}, \tag{12}$$