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Nonlinear dynamics of a rotating double pendulum

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ABSTRACT

Nonlinear dynamics of a double pendulum rotating at a constant speed about a vertical axis passing through the top hinge is investigated. Transitions of oscillations from chaotic to quasiperiodic and back to chaotic again are observed with increasing speed of rotation. With increasing speed, a pair of new stable equilibrium states, different from the normal vertical one, appear and the quasiperiodic oscillations occur. These oscillations are first centered around the origin, but with increasing rotation speed they cover the origin and the new fixed points. At a still higher speed, more than one pair of fixed points appear and the oscillation again turns chaotic. The onset of chaos is explained in terms of internal resonance. Analytical and numerical results confirm the critical values of the speed parameter at various transitions.

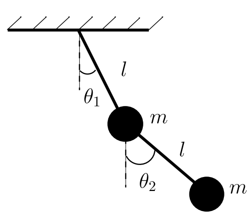

Fig. 1. A standard double pendulum.

The double pendulum is one of the primary examples of studying Hamiltonian chaos [1]. It is a conserved but non-integrable system having only one conserved quantity instead of two. The system is shown in Fig. 1 in the special case of the two pendulums having the same length and mass. The small amplitude motion is quasi-periodic in general and the high energy motion which is dominated entirely by the kinetic energy (the maximum potential energy is $6mgl$ if the zero is taken as the freely hanging position) with $g \to 0$, is once again integrable [2]. Chaos in the system has been studied primarily for E slightly greater than $6mgl$ (when both pendula are capable of undergoing full rotation) [2]. It was shown recently that chaos can exist in the system, when the energy is much lower ($E \gtrsim 2mgl$, only the outer pendulum can rotate), but this is strongly dependent on initial conditions. It can be argued that in principle, *Kolmogorov–Arnold–Moser theorem* will imply initial condition dependent breakdown of tori, but the dependence of the low energy chaos in the double pendulum is more systematic [3]. The system has two normal modes — "in-phase" and "out-of-phase". It is the out-of-phase initial conditions that predominantly lead to chaos at low energies. This can be linked to a mode softening of the "out-of-phase" mode. In another very well studied non-integrable conservative system — the extensible pendulum (also known as spring pendulum), an intriguing phenomenon was studied two decades ago [4–7]. This was a sequence of order–chaos–order transition. This problem is slightly simpler to deal with because apart from the energy, there is yet another parameter in the problem, the ratio of frequencies of the two normal modes of the system — the frequency ω_1 of the spring vibration and the frequency ω_2 of the pendulum motion. The system becomes resonant when $\frac{\omega_1}{\omega_2}$ is an integer $\geqslant 2$. The resonant condition leads to chaotic motion in the vicinity causing the order chaos transition. It is clear that if the ratio is very large, the spring is very stiff and is just a perturbative influence on the pendulum motion causing the overlapping resonance to disappear. Thus order is regained. The lesson to be learnt from this interesting problem is that a second parameter can play an important role and as we will see a vital role. With this in mind, we introduce a second parameter in the double pendulum via uniform rotation of the system as whole about a vertical axis passing through the top hinge. We consider the planar oscillation of the pendulum as seen by an observer rotating with the pendulum. The only other work on this double

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pendulum was carried out by Bridges and Georgiou [8], who concentrated on the motions in two different planes. We find over here for the low energies ($E \gtrsim 3mgl$) the chaos inducing "out-of-plane" initial conditions provide a chaos–order–chaos sequence as the rotation rate is increased.

The Lagrangian of a double pendulum can be written as,

$$L = ml^2\dot{\theta}_1{}^2 + \frac{1}{2}ml^2\dot{\theta}_2{}^2 + ml^2\dot{\theta}_1\dot{\theta}_2 cos(\theta_1 - \theta_2) + 2mgl(\cos\theta_1 - 1) + mgl(\cos\theta_2 - 1) \tag{1}$$

Here a constant ($3mgl$) has been deliberately added to make the total energy zero at the resting position. It is important to introduce directly the normal modes X_1 and X_2 through the relations

$$\theta_1 = \frac{1}{2}(X_1 + X_2) \tag{2a}$$

$$\theta_2 = \frac{1}{\sqrt{2}}(X_1 - X_2) \tag{2b}$$

Our experience with the dominant terms of the Lagrangian for intermediate amplitude motion leads to an effective Lagrangian

$$L = ml^2[\frac{1}{2}\dot{X}_1^2(1 + \frac{1}{\sqrt{2}}) + \frac{1}{2}\dot{X}_2^2(1 - \frac{1}{\sqrt{2}}) - \frac{\Omega^2}{2}(X_1^2 + X_2^2) - \frac{(\dot{X}_1^2 - \dot{X}_2^2)}{32\sqrt{2}}\{(2+\sqrt{2})X_2 - (2-\sqrt{2})^2X_1\}^2 + \ \ldots] \tag{3}$$

where, $\Omega^2 = \frac{g}{l}$. It is clear from Eq. (1) that X_1 is the "in-phase" mode and X_2 is the "out-of-phase" mode and from Eq. (3) that the frequency associated with X_1 is $(2-\sqrt{2})^{\frac{1}{2}}\Omega$ and that with X_2 is $(2+\sqrt{2})^{\frac{1}{2}}\Omega$. The resulting equation of motion (retaining terms of cubic order non-linearity only) for the mode under scrutiny, X_2, is

$$\ddot{X}_2 + (2+\sqrt{2})\Omega^2 X_2 = \frac{(2+\sqrt{2})\Omega^2 X_2}{16}(\sqrt{2}+1)[(2+\sqrt{2})X_2 - (2-\sqrt{2})X_1]^2 - \frac{(2+\sqrt{2})^2\dot{X}_2{}^2X_2}{16\sqrt{2}} + \frac{(2+\sqrt{2})\dot{X}_2{}^2X_2}{32\sqrt{2}} \tag{4}$$

The effective frequency for this mode X_2 is $[2+\sqrt{2}-4A^2]^{\frac{1}{2}}\Omega$ (where A is the amplitude of X_2 mode). Here, the X_2^3 and $\dot{X}_2{}^2X_2$ terms combine to give the strong mode softening from finite amplitude effect. The equation for the mode X_1 can similarly be written down and it just happens that the amplitude dependence of the "in-phase" mode is negligible in the first approximation and for our purpose, we will assume $X_1 = f(t)$, where $f(t)$ is some periodic function with basic frequency $(2-\sqrt{2})^{\frac{1}{2}}\Omega$. We note that the frequency ratio of the "in-phase" and "out-of-phase" modes is an irrational number $(3-2\sqrt{2})$ and hence looking for resonance is far more complicated with the double pendulum. The equation of motion (Eq. (4)) and can be viewed as the equation of motion of a one dimensional system which can always be given a Hamiltonian description (the relevant Lagrangian can be obtained from Eq. (3) by letting $X_1 = \dot{X}_1 = 0$ and the Hamiltonian will follow by the standard prescription). The coupling of X_2 to X_1 will be modeled by keeping the term $\dot{X}_2{}^2X_2X_1$ in Eq. (3) and we will arrive at a Hamiltonian whose structure for the motion of X_2

$$H = H_0(p, X_2) + H'(p, X_2, X_1(t)) \tag{5}$$

The structure of H' having been explained, it is clear that the resonance can happen if the frequency $(2-\sqrt{2})^{\frac{1}{2}}\Omega$ equals the effective frequency $[2+\sqrt{2}-4A^2]^{\frac{1}{2}}\Omega$ of the X_2 mode, A being the amplitude of the mode X_2. This leads to $A^2 = \frac{1}{\sqrt{2}}$. This clearly

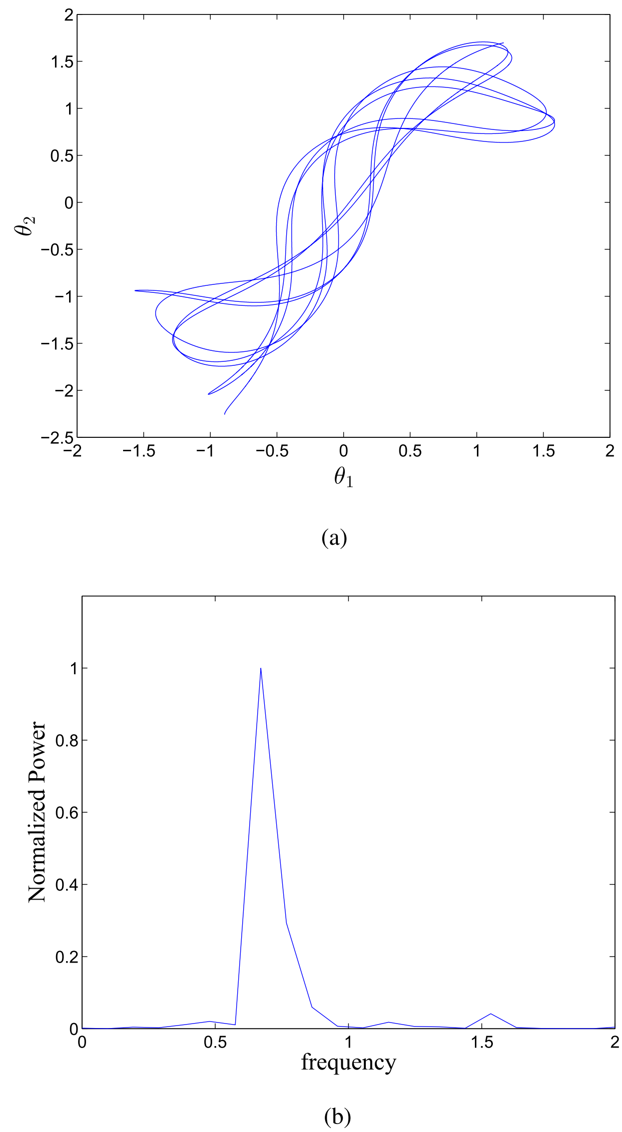

Fig. 2. (a) Quasi-periodic trajectory, (b) frequency spectrum for $\theta_1^{(0)} = 1.2$ and $\theta_2^{(0)} = 1.2\sqrt{2}$.

shows that it is possible to have resonance at low energies and that opens up the possibility of wide-spread chaos at relatively low energies.

We show the relevant results from numerics in Figs. 2–4. In Fig. 2a, we show a quasi-periodic trajectory for $\theta_1^{(0)} = 1.2$ and $\theta_2^{(0)} = 1.2\sqrt{2}$. The two incommensurate frequencies of the motion can be read off from the Fourier Transform shown in Fig. 2b. All frequencies are divided by Ω. In Fig. 3, we show how the frequencies change with energy for pure normal mode initial conditions. In Fig. 4, we show a typical chaotic trajectory at $E \simeq 2.4mgl$.

We differentiate between chaotic and ordered trajectories by calculating Largest Lyapunov Exponent (LLE). The computation of Lyapunov spectrum is done using the algorithm due to Wolf et al. [9]. Two trajectories are considered with and initial separation of a very small interval R_0. The first trajectory is called the fiducial trajectory and the second is called the perturbed trajectory. Both are followed together until the separation $|R_1 - R_0|$ is large enough and the LLE can be estimated as $\lambda = \frac{1}{\Delta t}\ln\frac{|R_1|}{|R_0|}$. The perturbed trajectory is then moved back to a separation R_0 along the vector joining the two end points and the process is repeated N times. The average over the N steps gives the Largest Lyapunov Exponent.