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Displaced logarithmic profile of the velocity distribution in the boundary layer of a turbulent flow over an unbounded flat surface

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ABSTRACT

It is shown that the Reynolds equations for a turbulent flow over an unbounded flat surface in the presence of a constant pressure-gradient lead to a displaced logarithmic profile of the velocity distribution; the displaced logarithmic profile is obtained by assuming a constant production rate of turbulence energy. The displacement height measured on the (vertical) axis perpendicular to the surface is either positive or negative. For a positive displacement height the boundary layer exhibits an inversion, while for a negative displacement height the boundary layer is a direct one. In an inversion boundary layer the logarithmic velocity profile is disrupted into two distinct branches separated by a logarithmic singularity. The viscosity transforms this logarithmic singularity into a sharp edge, governed by a generalized Reynolds number. The associated temperature distribution is calculated, and the results are discussed in relation to meteorological boundary-layer jets and stratified layers. The effects of gravitation and atmospheric thermal or fluid-mixture concentration gradients ("external forcings") are also considered; it is shown that such circumstances may lead to various modifications of the boundary layers. A brief presentation of a similar situation is described for a circular pipe.

Usually, the velocity logarithmic profile in the boundary layer of a turbulent flow over an unbounded flat surface is derived by using dimensional or similarity arguments [1,2]. We examine here the implications of the Reynolds equations for the turbulent boundary layer in the presence of a pressure gradient. It is shown that a constant pressure-gradient leads to a linear dependence of the Reynolds shear stress on the distance from the surface; such a dependence, combined with the assumption of a constant production rate of turbulence energy, yields a displaced logarithmic profile of velocity. The displacement height, measured along the (vertical) axis perpendicular to the surface, is either positive or negative, corresponding to an inversion or a direct boundary layer, respectively. The difference between the two types of boundary layers arises from the boundary conditions. In the inversion layer the fluid flows in the direction opposite to the main flow, and the logarithmic law of the velocity profile is splitted into two branches separated by a logarithmic singularity. The viscosity transforms this singularity into a sharp edge, governed by a generalized Reynolds number. The temperature distribution associated with such boundary layers is calculated, and the results are discussed in connection with the meteorological boundary-layer jets and stratified layers. Gravitation and atmospheric thermal or fluid-mixture concentration gradients ("external forcings") are also considered; it is shown that such circumstances may lead to various modifications of the boundary layers. A similar situation is presented for a circular pipe.

Specifically, we are interested in a turbulent flow of an incompressible fluid along an infinite plane surface. The coordinates x and y lie on the surface and the coordinate z is perpendicular to the surface (vertical coordinate). The velocity components (u, v, w) correspond to the (x, y, z)-directions. The fluid flows along the x axis with velocity u. As usually, we introduce the mean velocities $\bar{u}$, $\bar{v}$ and $\overline{w}$ and the fluctuating velocities u', v' and w', by $u \to \bar{u} + u'$, etc. (we consider time averaging; also, spatial averaging will be discussed below). We assume $\bar{v} = \overline{w} = 0$ and $\bar{u}(z) \neq 0$ depending only on z (a uniform flow along the x-direction). Under these conditions the Navier–Stokes equations lead to the Reynolds equations [1,3,4]

$$\begin{aligned} 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}\overline{u'^2} - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} + \nu\frac{\partial^2 \bar{u}}{\partial z^2} , \\ 0 &= -\frac{\partial}{\partial x}\overline{u'v'} - \frac{\partial}{\partial y}\overline{v'^2} - \frac{\partial}{\partial z}\overline{v'w'} , \\ 0 &= -\frac{\partial}{\partial x}\overline{u'w'} - \frac{\partial}{\partial y}\overline{v'w'} - \frac{\partial}{\partial z}\overline{w'^2} , \end{aligned} \quad (1)$$

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where ρ is the fluid density, p is the (mean) pressure (depending only on x) and ν is the viscosity coefficient. We note the occurrence in equation (1) of the correlation functions $\overline{u'^2}$, $\overline{u'v'}$, etc. (also called variances, like $\overline{u'^2}$, and covariances, like $\overline{u'v'}$); these quantities are the components of the Reynolds stress tensor (which, multiplied by ρ, is a momentum flux density). We assume a constant, negative pressure-gradient $\partial p/\partial x = const < 0$. We note that the Reynolds stress tensor generates forces which may compete with the pressure-gradient force in equation (1) (and with the viscosity "force"); therefore, equation (1) is in fact an equilibrium equation (corresponding to a steady flow), as expected.

Equations (1) represent a system of three equations with seven unknowns: the components of the Reynolds tensor and the mean velocity $\bar{u}$; it is an under-determined system of equations. We are interested in the first equation (1), where we assume $\partial(\overline{u'v'})/\partial y = 0$, $\overline{u'w'}(z) \neq 0$ depending only on z and $\partial\overline{u'^2}/\partial x = const$. In general, constant Reynolds stress components with respect to a coordinate amount to a homogeneous turbulence along that axis [5–7]. Under these conditions, the first equation (1) reads

$$0 = A - \frac{d}{dz}\overline{u'w'} + \nu\frac{d^2\bar{u}}{dz^2} \ , \quad (2)$$

where $A = -(1/\rho)dp/dx - \partial\overline{u'^2}/\partial x$; for the sake of generality we keep for the moment $\partial\overline{u'^2}/\partial x = const$ in equation (2), corresponding to an inhomogeneous turbulence along the x-axis.

We leave aside for the moment the viscosity term in equation (2); then, the integration of this equation gives

$$\overline{u'w'} = Az - \beta u_*^2 \ , \quad (3)$$

where $\beta = \pm 1$ and the parameter u_* is a surface friction velocity; for the sake of generality we keep both signs in the boundary condition $\overline{u'w'}\mid_{z=0} = -\beta u_*^2$; ρu_*^2 is the friction force per unit area of the surface and $\rho\overline{u'w'}$ is the xz-component of the momentum flux density (Reynolds shear stress). Equation (3) can also be written as

$$\overline{u'w'} = \frac{\beta u_*^2}{h}(z-h) \ , \ h = \frac{\beta u_*^2}{A} \ , \ A \neq 0. \quad (4)$$

Such a linear dependence of the shear stress is known in the atmospheric turbulence of the boundary layers [8] and in turbulent flow on flat plates or in channels [1]. We note that the displacement height h may have both signs.

Multiplying the Navier–Stokes equations by $\bar{u}$ and using the same procedure ($u \to \bar{u} + u'$, etc.), we get the conservation law for the mean-flow energy

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\overline{\left[\frac{1}{2}\bar{u}^2\right]} = \\ &= \bar{u}\left[-\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}\overline{u'^2} - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} + \nu\frac{\partial^2\bar{u}}{\partial z^2}\right] \ ,\end{aligned} \quad (5)$$

which is the first equation (1) multiplied by $\bar{u}$. Similarly, multiplying the Navier–Stokes equations by the fluctuating velocities and taking the average we get the conservation equation for the turbulence energy

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\overline{\left[\tfrac{1}{2}(u'^2+v'^2+w'^2)\right]} = \\ &= -\overline{u'w'}\tfrac{\partial\bar{u}}{\partial z} - \tfrac{1}{2}\bar{u}\tfrac{\partial}{\partial x}\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right) + \\ &+ \nu\left(\overline{u'\Delta u'} + \overline{v'\Delta v'} + \overline{w'\Delta w'}\right) \ ,\end{aligned} \quad (6)$$

where third-order terms involving products of three fluctuating velocities and velocity derivatives, as well as the contribution of the fluctuating part of the pressure have been dropped out; in addition, in deriving equation (6) the continuity equation $\partial u'/\partial x + \partial v'/\partial y + \partial w'/\partial z = 0$ has been used. The main assumption made here is that the fluctuations are small in comparison with the mean flow. Adding the two equations (5) and (6), we get the conservation law of the total energy

$$\begin{aligned}0 &= \overline{\tfrac{\partial}{\partial t}\left[\tfrac{1}{2}(u^2+v^2+w^2)\right]} = \\ &= -\tfrac{1}{\rho}\tfrac{\partial p}{\partial x}\bar{u} - \tfrac{\partial}{\partial x}\left(\bar{u}\overline{u'^2}\right) - \tfrac{\partial}{\partial y}\left(\bar{u}\overline{u'v'}\right) - \\ &- \tfrac{\partial}{\partial z}\left(\bar{u}\overline{u'w'}\right) - \tfrac{\partial}{\partial x}\left[\tfrac{1}{2}\bar{u}\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right)\right] + \\ &+ \nu\left(\bar{u}\tfrac{\partial^2\bar{u}}{\partial z^2} + \overline{u'\Delta u'} + \overline{v'\Delta v'} + \overline{w'\Delta w'}\right) \ .\end{aligned} \quad (7)$$

The first term on the right in equation (7) is related to the work done by the pressure forces per unit time; the next four terms are related to the energy flux density due to the fluid mass transfer; the last term, involving the viscosity coefficient, can be written as

$$\begin{aligned}&\nu\left(\bar{u}\tfrac{\partial^2\bar{u}}{\partial z^2} + \overline{u'\Delta u'} + \overline{v'\Delta v'} + \overline{w'\Delta w'}\right) = \\ &= \nu\tfrac{\partial^2}{\partial z^2}\left(\tfrac{1}{2}\bar{u}^2\right) + \tfrac{1}{2}\nu\Delta\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right) - \\ &- \nu\left(\tfrac{\partial\bar{u}}{\partial z}\right)^2 - \nu\left[\overline{\left(\tfrac{\partial u'}{\partial x}\right)^2} + \overline{\left(\tfrac{\partial u'}{\partial y}\right)^2} + \overline{\left(\tfrac{\partial u'}{\partial z}\right)^2}\right] - \\ &- \nu\left[\overline{\left(\tfrac{\partial v'}{\partial x}\right)^2} + \overline{\left(\tfrac{\partial v'}{\partial y}\right)^2} + \overline{\left(\tfrac{\partial v'}{\partial z}\right)^2}\right] - \\ &- \nu\left[\overline{\left(\tfrac{\partial w'}{\partial x}\right)^2} + \overline{\left(\tfrac{\partial w'}{\partial y}\right)^2} + \overline{\left(\tfrac{\partial w'}{\partial z}\right)^2}\right] \ ;\end{aligned} \quad (8)$$

the first two terms on the right in equation (8) are related to the energy flux density due to internal friction (momentum transfer through collisions caused by viscosity); the remaining terms imply heat production; for an adiabatic flow they are equal to $-(1/\rho)\,\mathrm{div}\,\mathbf{q}$, where $\mathbf{q}$ is the heat flux density [2].

A similar analysis can be done for each of the equations (5) and (6) separately. The terms $-\bar{u}\frac{\partial}{\partial z}\overline{u'w'}$ on the right in equation (5) and $-\overline{u'w'}\frac{\partial\bar{u}}{\partial z}$ in equation (6) (which together give an energy flux density $-\frac{\partial}{\partial z}\left(\bar{u}\overline{u'w'}\right)$), have a special meaning when taken separately: each of them represents a coupling between the mean flow and the turbulent flow. In particular, $-\overline{u'w'}\frac{\partial\bar{u}}{\partial z}$ is a production rate of turbulence energy (while $-\bar{u}\frac{\partial}{\partial z}\overline{u'w'}$ is a production rate of mean-flow energy; "production" means here either a positive or a negative contribution).

We focus now on equation (6). The components of the Reynolds stress tensor have a small, local (finite) variation, at least for a homogeneous turbulence; assuming a homogeneous turbulence along the x, y-coordinates and taking a spatial averaging with respect to these coordinates we get from equation (6)

$$\begin{aligned}\overline{u'w'}\tfrac{d\bar{u}}{dz} &= \nu\left(\overline{u'\tfrac{d^2}{dz^2}u'} + \overline{v'\tfrac{d^2}{dz^2}v'} + \overline{w'\tfrac{d^2}{dz^2}w'}\right) - \nu C = \\ &= \tfrac{1}{2}\nu\tfrac{d^2}{dz^2}\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right) - \\ &- \nu\left[\overline{\left(\tfrac{du'}{dz}\right)^2} + \overline{\left(\tfrac{dv'}{dz}\right)^2} + \overline{\left(\tfrac{dw'}{dz}\right)^2}\right] - \nu C \ ,\end{aligned} \quad (9)$$

where C is a positive constant arising from spatial averages of the type $u'\frac{\partial^2}{\partial x^2}u'$ (for the sake of simplicity, in equation (9) the spatial averaging is not indicated explicitly by an additional averaging sign). The first term on the right in equation (9) involves