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Finiteness of integrable n-dimensional homogeneous polynomial potentials ☆

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Abstract

We consider natural Hamiltonian systems of $n > 1$ degrees of freedom with polynomial homogeneous potentials of degree k. We show that under a genericity assumption, for a fixed k, at most only a finite number of such systems is integrable. We also explain how to find explicit forms of these integrable potentials for small k.

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1. Introduction

At least half of the models which appear in physics, astronomy and other applied sciences have a form of a system of ordinary differential equations depending usually on several parameters. The question if the considered system possesses one or more first integrals is fundamental. First integrals give conservation laws for the model. Moreover, from an operational point of view, they simplify investigations of the system. In fact, we can always lower the dimension of the system by the number of its independent first integrals. If we know a sufficient number of first integrals, we can solve explicitly the considered system. As a rule, except possible obvious first integrals, as Hamiltonians for Hamilton's equations, additional first integrals exist only for specific values of parameters of the considered systems. Thus, the problem is how to find these values of parameters, or, how to show that the system does not admit any additional first integral for specific values of the parameters. The problems mentioned above are generally very hard, and in spite of their basic physical importance there are no universal methods to solve them even for very special classes of differential equations.

In past the search for first integrals was based on the direct method due to Darboux, see e.g. [1]. Applying this method, we postulate a general form of the first integral. Usually, this first integral depends on some unknown functions. The condition that it is constant along solutions of the analysed system gives rise to a set of partial differential equations determining the unknown functions. Complexity of the obtained partial differential equations is the reason why it is usually assumed that the first integral is a polynomial with respect to momenta of low degree. For more information about the direct method see [2].

In the sixties of the previous century, Ablowitz, Ramani and Segur [3,4] proposed a completely different method of searching for integrable systems. The kernel of this method, originating from the works of Kovalevskaya [5,6] and Painlevé [7], is a conjecture that solutions of integrable systems after the ex-

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tension to the complex time plane should be still simple, more precisely, single-valued. If all solutions of a given system are single-valued, then we say that it possesses the Painlevé property. But, to check if a given system possesses the Painlevé property, we must have at our disposal a single-valued particular solution of the system (or its appropriate truncation). Then the necessary condition for the Painlevé property is following: all solutions of the variational equations along this single-valued particular solution are single-valued. If for some specific values of parameters the considered system has the Painlevé property, then, assuming those values of parameters, we can look for first integrals applying the direct method. This means that the Painlevé property has played the role of necessary integrability conditions and, for this reason, it is sometimes called the Painlevé test. The results of Kovalevskaya and Lyapunov [5,6,8] showed that checking the Painlevé property is in fact reduced to checking if a certain matrix, the so-called Kovalevskaya matrix (see the next section), is semisimple, and its eigenvalues, the so-called Kovalevskaya exponents, are integers. Yoshida [9] showed that Kovalevskaya exponents are related to the degrees of first integrals, and this fact simplifies the second step of the analysis, namely, finding the explicit forms of the fist integrals. The Painlevé test appeared to be very effective and many new integrable systems were found thanks to its application. The main advantage of this method is its simplicity. Its weak point is the fact that there is no rigorous proof that the Painlevé property is directly related to the integrability. In fact, there are known examples of integrable systems that do not pass the Painlevé test, by this reason the weak Painlevé test was introduced, see [10–12].

Let us remark that in Hamiltonian mechanics there exist a few other tools for testing the integrability, see [13], however, they usually work for very restricted classes of Hamiltonian systems.

Quite recently two mathematically rigorous approaches to the integrability problem formulated by Ziglin [14,15] and Morales-Ruiz and Ramis [16,17] have appeared. They explain relations between the existence of first integrals and branching of solutions as functions of the complex time and give necessary integrability conditions for Hamiltonian systems. It appears that the integrability is related to properties of the monodromy group or the differential Galois group of variational equations along a particular solution.

In this Letter we apply the Morales-Ruiz–Ramis approach to the Hamiltonian systems defined in a linear symplectic space, e.g., $\mathbb{R}^{2n}$ or $\mathbb{C}^{2n}$ equipped with canonical variables $\boldsymbol{q} = (q_1, \dots, q_n)$, $\boldsymbol{p} = (p_1, \dots, p_n)$, and given by a natural Hamiltonian function

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}). \tag{1}$$

We assume that $V(\boldsymbol{q})$ is a homogeneous polynomial of degree $k > 2$. The integrability of Hamiltonian systems with Hamiltonian (1) was analysed by the direct method, the Painlevé analysis and some other techniques, see [2,11,18,19]. Nevertheless, a quick overview of the literature shows that except for some "easy" cases only sporadic examples of integrable systems with two or three degrees of freedom governed by the Hamiltonian of the form (1) were found. In all integrable cases first integrals are polynomials and their degrees with respect to the momenta are not greather than four. Hence, it is natural to ask: do we know all integrable systems with Hamiltonian (1)? It is hard to believe that the answer to this question is positive. In fact, as far as we know, in all works only very limited families of such systems were investigated. Thus, what can we expect? Are there infinitely many integrable Hamiltonian systems which wait to be discovered?

The aim of this note is to give a necessarily limited answer to the above question. The main result of this Letter shows that assuming that potential V is generic, the number of meromorphically integrable systems with Hamiltonian (1) is finite.

Let us explain here what does it mean a generic potential. Hamilton's equations generated by (1) admit particular solutions of the form

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d}, \qquad \boldsymbol{p}(t) = \dot{\varphi}(t)\boldsymbol{d}, \tag{2}$$

provided $\boldsymbol{d}$ is a nonzero solution of

$$V'(\boldsymbol{d}) = \boldsymbol{d}, \tag{3}$$

and $\varphi(t)$ satisfies $\ddot{\varphi} = -\varphi^{k-1}$. A direction $\boldsymbol{d} \in \mathbb{C}^n$ defined by a solution of (3) is called a Darboux point of potential V. We say that potential V is generic iff it admits exactly $[(k-1)^n - 1]/(k-2)$ different Darboux points. For details see Section 2.

To prove our finiteness result we combine the Morales-Ruiz–Ramis theory and a kind of global Kovalevskaya analysis of the auxiliary system

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} = V'(\boldsymbol{q}). \tag{4}$$

It appears that the Kovalevskaya exponents of the above system are closely related to the integrability of Hamiltonian system given by (1). The Morales-Ruiz–Ramis theory gives strong restrictions on their values. On the other hand, we can calculate the Kovalevskaya exponents for different particular solutions of (4). The key point is the fact that the Kovalevskaya exponents calculated for different solutions are not arbitrary, i.e., there exist certain relations among them.

Just to avoid missunderstanding let us fix terminology here. We consider complex Hamiltonian systems with phase space $\mathbb{C}^{2n}$ equipped with the standard canonical structure. First integrals are always assumed to be meromorphic in appropriate domains. By saying that a potential V is integrable, we understand that the Hamilton equations generated by Hamiltonian (1) are integrable in the Liouville sense. It is easy to check that if potential $V(\boldsymbol{q})$ is integrable, then also $V_{\boldsymbol{A}}(\boldsymbol{q}) := V(\boldsymbol{A}\boldsymbol{q})$ is integrable for an arbitrary $\boldsymbol{A} \in \mathrm{GL}(n, \mathbb{C})$ satisfying $\boldsymbol{A}\boldsymbol{A}^T = \alpha\boldsymbol{E}$, $\alpha \in \mathbb{C}^\star$ and $\boldsymbol{E}$ is the identity matrix, see e.g. [2]. Potentials V and $V_{\boldsymbol{A}}$ are called equivalent, and the set of all potentials is divided into disjoint classes of equivalent potentials. Later a potential means a class of equivalent potentials in the above sense.

The plan of this Letter is following. In the next section we briefly recall basic facts from the Kovalevskaya analysis and