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Effects of network structure on the synchronizability of nonlinearly coupled Hindmarsh–Rose neurons

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ABSTRACT

This work is devoted to investigate the effects of network structure on the synchronizability of nonlinearly coupled dynamical network of Hindmarsh–Rose neurons with a sigmoidal coupling function. We mainly focus on the networks that exhibit the small-world character or scale-free property. By checking the first nonzero eigenvalue of the outer-coupling matrix, which is closely related to the synchronization threshold, the synchronizabilities of three specific network ensembles with prescribed network structures are compared. Interestingly, we find that networks with more connections will not necessarily result in better synchronizability.

1. Introduction

It is well known that synchronization is one of simplest and most prominent collective behaviors appearing in various biological and physical systems. For example, in many biological neural networks, it has been observed that coupling allows neurons to synchronize each other [1–7]. Moreover, many brain disorders such as Alzheimer's disease, epilepsy, Parkinson's disease, and schizophrenia have been linked to the abnormal patterns of synchronization in the brain [8–10]. Thus, understanding neuronal synchrony is one of the fundamental issues in neuroscience. In considering the mathematical models, it has been shown that the single Hindmarsh–Rose neuron is capable of producing major neural behavior such as spiking, bursting, and chaotic behavior. Therefore, to better understand the relation between the connectivity patterns and synchronization behavior, in this paper, we will consider the coupled Hindmarsh–Rose neurons and then compare the synchronizability of some specific neural networks with different coupling topologies.

Generally speaking, the study of synchronization in coupled dynamical networks can be categorized into two different aspects. The first one is to derive criteria that either ensure the coupled dynamical networks to be locally or globally synchronized or serve as necessary conditions for synchronization. To this aim, a variety of methods have been proposed such as the master stability function method [11], the connection graph stability method [12–14], the linear matrix inequality [15–18], the matrix measure approach [19–21] and the eigenvalue-based approach [22–24]. Most of these studies are predominately concentrated on the oscillators which are interconnected using linear coupling functions. In the study of neural networks, linear coupling corresponds to the neurons that are connected by electrical synapses. The electrical synapse is the result of the potential difference between the neurons and causes an immediate physiological response of the post-synaptic side. However, it has been shown that neurons can also be interconnected via chemical synapses and the chemical interaction is described by a nonlinear coupling function [25]. Thus, the study of synchronization in nonlinearly coupled dynamical networks is always an important and ongoing issue [15,16,20,22–24,26–30].

The second aspect of analyzing synchronization focuses on whether the network properties can have any influence on the network synchronizability. For example, earlier studies have pointed out two important features affecting the network synchronizability, the small-world effect and the scale-free property. The small-world effect means that the networks have a small average network distance as random networks and a large clustering coefficient as regular ones [31]. Networks with the scale-free property show a power law degree distribution [32]. Here, the average network distance, clustering coefficient, and degree distribution are the struc-

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tural properties of the networks. Apparently, there are also other factors [33] such as the average degree, betweenness centrality, etc., which may affect the network synchronizability. Some studies have shown that small-world and scale-free networks can significantly improve the synchronizability, compared with regular ones (see for instance [34–37]). For the degree distribution, it has been observed in [38] that increasing the degree heterogeneity often reduces the average network distance but may suppress synchronizability. However, Hong et al. [39] reported some certain cases for better synchronizability as the heterogeneity of the degree distribution is increased. The effect of clustering coefficient on the synchronizability of certain type of coupled dynamical networks has also been considered in [40]. The authors suggested that the more the network is clustered, the poorer the synchronizability is. Making mention of the betweenness centrality on nodes, Hong et al. [39] have found a consistent trend between the betweenness centrality and the network synchronizability.

As we can see in aforementioned studies, the correlation between the network synchronizability and network properties is rather complicated. We recognize that even networks with the same global properties, they may have very different synchronizability (see [41–43] for more details). Thus, the main purpose of this paper is to further explore how network properties, such as the average network distance and the node betweenness centrality, affect the global synchronizability of the network of nonlinearly coupled Hindmarsh–Rose neurons with a sigmoidal coupling function. We will mainly focus on the networks that exhibit the small-world character or scale-free property. Based on the synchronization criterion developed in [22], which is closely related to the first nonzero eigenvalue of the outer-coupling matrix, we investigate the synchronizability of several network ensembles in which networks have the same global properties. We examine various networks such as the modular networks [44], the small-world networks [31] and the scale-free networks [32]. More specifically, we first compare the synchronizability of three specific network ensembles consisting of networks with the same size and the same number of edges. The study reveals that for these three specific network ensembles, a small average network distance seems in favor of the global synchronizability of networks. We then investigate three specific network ensembles consisting of networks with the same size and the same average network distance. We find that in this case, a small value of the normalized betweenness centrality on nodes seems favorable for the network synchronizability and moreover, networks with more connections (or equivalently, with larger average degree) will not necessarily result in better synchronizability. This observation is consistent with the finding reported in [42], where several examples have provided to show that adding new edges to a network can either increase or decrease the network synchronizability.

The remainder of this paper is organized as follows. In Section 2, we first introduce the nonlinearly coupled dynamical network of Hindmarsh–Rose neurons with a sigmoidal coupling function, and then briefly review the synchronization criterion developed in [22]. In Section 3, we compare the synchronizability of three specific network ensembles that have the same global network properties. Finally, we make some conclusions in Section 4.

2. The nonlinearly coupled dynamical network of Hindmarsh–Rose neurons

We consider the following nonlinearly coupled dynamical network of m identical Hindmarsh–Rose neurons [45]:

$$\begin{cases} x_i'(t) = y_i - a x_i^3 + b x_i^2 - z_i + I \\ \qquad + c \displaystyle\sum_{j=1, j\neq i}^{m} a_{ij}\Big(g(x_j) - g(x_i)\Big), \\ y_i'(t) = 1 - d x_i^2 - y_i, \\ z_i'(t) = r(s(x_i + x_0) - z_i), \qquad i = 1, 2, \cdots, m, \end{cases} \tag{1}$$

where $\boldsymbol{x}_i(t) = (x_i(t), y_i(t), z_i(t))^\top \in \mathbb{R}^3$ is the state variable of the ith neuron; the intrinsic dynamics of ith neuron is described by the vector function f given by

$$f(\boldsymbol{x}_i) := \Big(y_i - a x_i^3 + b x_i^2 - z_i + I, \\ 1 - d x_i^2 - y_i, r(s(x_i + x_0) - z_i)\Big)^\top$$

with the positive parameters a, b, d, r, s, x_0 and I; the scalar $c > 0$ is the coupling strength; the real matrix $A = (a_{ij})_{m\times m}$, $a_{ij} \geq 0$ for all $i \neq j$, is the outer-coupling matrix representing the topological structure of the network. The nonlinear coupling function $g : \mathbb{R} \to \mathbb{R}$ is of sigmoidal type and satisfies the following properties:

$$\begin{cases} g \in C^2(\mathbb{R});\ \displaystyle\lim_{\xi\to\pm\infty} g(\xi) = u^{\pm} \in \mathbb{R}; \\ \exists\, \widetilde{\xi} \in \mathbb{R} \text{ such that } g'(\widetilde{\xi}) \geq g'(\xi) > 0 \text{ and} \\ \quad g''(\xi)\xi < 0 \text{ for all } \xi \in \mathbb{R}, \xi \neq \widetilde{\xi}. \end{cases} \tag{2}$$

A typical example satisfying this setting is given by $g(\xi) = \tanh(\xi)$ with $u^+ = 1$, $u^- = -1$, and $\widetilde{\xi} = 0$. Since $g(x_j) - g(x_i) = 0$ for $1 \leq j = i \leq m$, the value a_{ii} could be arbitrary in the coupled dynamical network (1). In this paper, we set $a_{ii} = -\sum_{j=1, j\neq i}^{m} a_{ij}$ for $1 \leq i \leq m$ and let $\Gamma = (1, 0, 0)^\top$, then the coupled system (1) can be expressed as

$$\boldsymbol{x}_i'(t) = f(\boldsymbol{x}_i(t)) + c\sum_{j=1}^{m} a_{ij}\Gamma g(x_j(t)), \quad i = 1, 2, \cdots, m. \tag{3}$$

In our recent work [22], we have developed a sufficient condition to ensure the global exponential synchronization of the nonlinearly coupled dynamical network (3) of Hindmarsh–Rose neurons. Here, the coupled dynamical network (3) is said to be globally exponentially synchronized if there exist constants $\varepsilon > 0$ and $T > 0$ such that for any initial values $(\boldsymbol{x}_1(0), \boldsymbol{x}_2(0), \cdots, \boldsymbol{x}_m(0))^\top$, we have $\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| \leq \delta_0 e^{-\varepsilon t}$ for all $t > T$ and $1 \leq i, j \leq m$, where constant $\delta_0 > 0$ may depend on the initial values. To describe the sufficient condition, we need to make an assumption on the outer-coupling matrix A as follows. Let $w_i > 0$ for $1 \leq i \leq m$ be some real numbers such that $\sum_{i=1}^{m} w_i = 1$. We define an $m \times m$ diagonal matrix by $W = \text{diag}\{w_1, w_2, \cdots, w_m\}$.

Assumption (H). The outer-coupling matrix A is an irreducible $m \times m$ real matrix in the form, $A = W^{-1}B$, where $B = (b_{ij})_{m\times m}$ is a symmetric matrix satisfying $b_{ij} = b_{ji} \geq 0$ for $i \neq j$ and $b_{ii} = -\sum_{j=1, j\neq i}^{m} b_{ij}$ for $i = 1, 2, \cdots, m$.

With Assumption (H), we have shown in [22] that the nonlinearly coupled dynamical network (3) is eventually dissipative, that is, all the solutions of the coupled system are eventually bounded by a compact set. In other words, there exists a time $T > 0$ and a compact set $\mathcal{B}$ such that every solution $(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_m(t))$ of (3) will eventually enter into $\mathcal{B}$ and stay there for all $t \geq T$. We then define the number

$$\ell := \min_{1\leq i\leq m}\{g'(x_i) : (\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_m(t)) \in \mathcal{B}\} > 0. \tag{4}$$