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# The even and odd supersymmetric Hunter–Saxton and Liouville equations

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ABSTRACT

It is shown that two different supersymmetric extensions of the Harry Dym equation lead to two different negative hierarchies of the supersymmetric integrable equations. While the first one yields the known even supersymmetric Hunter–Saxton equation, the second one is a new odd supersymmetric Hunter–Saxton equation. It is further proved that these two supersymmetric extensions of the Hunter–Saxton equation are reciprocally transformed to two different supersymmetric extensions of the Liouville equation.



## 1. Introduction

The list of unusual behavior of the supersymmetric integrable systems is rather long. It appears that during the supersymmetrization of the classical systems, some typical supersymmetric effects, compared with the classical theory, occur. The non-uniqueness of the roots of the supersymmetric Lax operator [13], the lack of the bosonic reduction to the classical equations [9] and the occurrence of the non-local conservation laws [8,6] have been discovered in the last century. Recently it appeared that both even and odd Hamiltonian operators could be used to the supersymmetrization of the classical integrable systems [2,11,12,15]. These effects rely strongly on the descriptions of the generalized classical systems which we would like to supersymmetrize.

In this Letter we prolong this list, namely we show that the Hunter–Saxton (HS) equation, similarly as the Harry Dym (HD) equation, could be supersymmetrized in two different manners. It is known that the classical HD equation is supersymmetrized in two ways, either by even or by odd supersymmetric Hamiltonian operators [2,11]. It should be remarked that in addition to these two cases, a super HD equation was deduced from the fermionic extension of energy-dependent Schrödinger operator [1].

The even supersymmetric HD equation is a multi-Hamiltonian system and its negative hierarchy contains the even supersymmetric generalization of the HS equation. The second Hamiltonian structure for this extension is generated by the supersymmetric centerless Virasoro algebra. In the second approach, where we use the odd Hamiltonian operators, an odd supersymmetric HD equation is obtained from the Lax representation. However this equation admits bi-Hamiltonian formulation only and does not possess the 'first' Hamiltonian operator.

As we shall show that, from the knowledge of the second and third odd supersymmetric Hamiltonian structures of the odd supersymmetric HD equation, it is possible to construct new negative hierarchy of equations. The flows of this hierarchy contain a new supersymmetric extension of the HS equation.

It is well known that the classical HS equation under reciprocal transformation reduces to the Liouville equation [5]. We show that it is possible to apply the supersymmetric analogue of reciprocal transformation described in [11] to two different supersymmetric extensions of the HS equation, and as a result two supersymmetric extensions of the Liouville equation are obtained.

The Letter is organized as follows. In the first section we recapitulate the known facts on the bi-Hamiltonian formulation of the classical HS equation and explain its connections with the negative hierarchy of the HD equation. In the next section the even hierarchy of the supersymmetric HD equation as well as its negative hierarchy, which contains the even supersymmetric extension of the HS equation, are presented. Also this section describes the reciprocal link between the even supersymmetric HS equation and a new supersymmetric Liouville equation. The third section contains our main result where the negative hierarchy of the odd supersymmetric HD equation is introduced. This hierarchy takes an odd supersymmetric HS equation as one of its flows. Similar to the previous section we describe a reciprocal link between this odd supersymmetric HS equation and a new supersymmetric Liouville equation. In Appendix A a simple proof

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is presented for the fact that the third Hamiltonian operators of the even as well odd supersymmetric HD equations do satisfy the Jacobi identity.

## 2. Hunter–Saxton equation

The Hunter–Saxton equation [7]

$$u_{xxt} = 2u_x u_{xx} + u u_{xxx} \tag{1}$$

could be constructed in two different approaches. In the first approach this equation is considered as the special limit of the Camassa–Holm equation [5] while in the second one it is regarded as a member of the negative hierarchy of the Harry Dym equation.

Let us briefly recapitulate these scenarios.

Both the HS equation and the Camassa–Holm equation could be embedded into the general model

$$\lambda u_t - u_{xxt} = \frac{1}{2}\left(-3\lambda u^2 + 2u u_{xx} + u_x^2\right)_x, \tag{2}$$

where $\lambda$ is an arbitrary parameter. For $\lambda = 1$ Eq. (2) is just the Camassa–Holm equation [3,4] while for $\lambda = 0$ it becomes the HS equation (1).

The general equation (2) is a bi-Hamiltonian system

$$m_t = K_1 \frac{\delta H_1}{\delta m} = K_2 \frac{\delta H_2}{\delta m},$$

where $m = \lambda u - u_{xx}$ and

$$H_1 = \frac{1}{2}\int \mathrm{d}x\, u m, \qquad H_2 = -\frac{1}{2}\int \mathrm{d}x\left(\lambda u^3 + u u_x^2\right),$$
$$K_1 = -(\partial_x m + m\partial_x), \qquad K_2 = \left(\lambda - \partial_x^2\right)\partial_x.$$

To put the HS equation into the negative HD hierarchy, we first consider the bi-Hamiltonian structure of the HD equation

$$w_t = P_1 \frac{\delta H_{-1}}{\delta w} = P_2 \frac{\delta H_{-2}}{\delta w} = \left(w^{-\frac{1}{2}}\right)_{xxx} \tag{3}$$

where

$$P_1 = \partial_x^3, \qquad P_2 = \partial_x w + w\partial_x = 2w^{\frac{1}{2}}\partial_x w^{\frac{1}{2}},$$
$$H_{-1} = 2\int \mathrm{d}x\, w^{\frac{1}{2}}, \qquad H_{-2} = \frac{1}{8}\int \mathrm{d}x\, w^{-\frac{5}{2}} w_x^2. \tag{4}$$

As we see in both approaches we deal with the same bi-Hamiltonian structure. In fact the HD equation possesses the multi-Hamiltonian structure which is constructed out of the recursion operator $R = P_2 P_1^{-1}$ and $P_2$ and reads as $P_{n+2} = R^n P_2, n = 1, 2, \ldots$. For example the third Hamiltonian structure is

$$w_t = P_3 \frac{\delta H_{-3}}{\delta w}$$

where

$$P_3 = P_2 P_1^{-1} P_2 = (\partial_x w + w\partial_x)\partial_x^{-3}(\partial_x w + w\partial_x),$$
$$H_{-3} = -\frac{1}{2}\int \mathrm{d}x\left(16 w_{xx}^2 w^{-\frac{7}{2}} - 35 w_x^4 w^{-\frac{11}{2}}\right).$$

Using the Hamiltonian operators $P_1$ and $P_2$, it is possible to construct the negative as well as positive hierarchies of flows. For the negative hierarchy, the first three flows are

$$w_{t_1} = P_1 \frac{\delta H_0}{\delta w} = P_2 \frac{\delta H_{-1}}{\delta w} = P_3 \frac{\delta H_{-2}}{\delta w} = 0,$$
$$w_{t_2} = P_1 \frac{\delta H_1}{\delta w} = P_2 \frac{\delta H_0}{\delta w} = w_x, \tag{5}$$
$$w_{t_3} = P_1 \frac{\delta H_2}{\delta w} = P_2 \frac{\delta H_1}{\delta w} = P_3 \frac{\delta H_0}{\delta w} = -w_x\left(\partial_x^{-2} w\right) - 2w\left(\partial_x^{-1} w\right), \tag{6}$$

where

$$H_0 = \int \mathrm{d}x\, w, \qquad H_1 = \frac{1}{2}\int \mathrm{d}x\left(\partial_x^{-1} w\right)^2, \qquad H_2 = \frac{1}{2}\int \mathrm{d}x\left(\partial_x^{-2} w\right)\left(\partial_x^{-1} w\right)^2. \tag{7}$$

Eq. (6) is the HS equation (1) after identifying $t_3 = -t$, $w = u_{xx}$. Notice that $w_{t_2}$ is not a tri-Hamiltonian because

$$P_3^{-1} w_x = \frac{1}{4} w^{-\frac{1}{2}}\partial_x^{-1} w^{-\frac{1}{2}}\partial_x^3 w^{-\frac{1}{2}}\partial_x^{-1} w^{-\frac{1}{2}} w_x = 0.$$