Physics Letters A 379 (2015) 650–653

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Material deformation tensor in time-reversal symmetry breaking turbulence

A.S. Il'yn [a], K.P. Zybin [a,b]

[a] P.N. Lebedev Institute of Physics, Theory Department, 119991 Moscow, Russia
[b] Higher School of Economics, Mathematical Department, 101000 Moscow, Russia

ARTICLE INFO

Article history:
Received 18 December 2014
Accepted 19 December 2014
Available online 23 December 2014
Communicated by V.M. Agranovich

ABSTRACT

The properties of material deformation tensor in time irreversible turbulence are determined. It is shown that time irreversibility in the Lagrangian framework is connected with energy flux of the turbulence.

1. Introduction

The Lagrangian evolution of material elements has been extensively studied in various works [1–3]. These studies were connected with different problems: such as Lagrangian turbulence [5] or passive scalar decay [2]. One of the most important results of these studies is the proof of the intermittency in structure functions. This intermittency seems closely connected with the intermittency of the developed turbulence. The results of these studies were summarized in [2]. But the most of theoretical papers consider hydrodynamic turbulence as a Gaussian noise. In the case of a Gaussian stochastic process there is time symmetry t to $-t$.

In real uniform and isotropic turbulence energy flows from the scale at which it is injected, L, to the scale where it is dissipated, η. For intense three-dimensional turbulence, $L \gg \eta$, and the energy flows from large to small scales [1]. As a result, time symmetry is broken, since the time reversal t to $-t$ would also reverse the direction of the energy flux. Exploring the implications of this time asymmetry on the relative motion between fluid particles is of great interest.

The simplest problem in this context concerns the dispersion of two particles whose positions, $r_1(t)$ and $r_2(t)$, are separated by $|r_2(t) - r_1(t)|$. The growth of the mean n-th power of the separation, forward ($t > 0$) and backward in time ($t < 0$) is a fundamental question in turbulence research [4].

The aim of this paper is an introduction of a model which takes into account time irreversibility and an investigation of the influence of this time asymmetry on material deformation tensor. Note that the time evolution of this tensor is a basis for the theories discussed above.

2. The statement of the problem

A description of material finite deformation in a continuum is done using the deformation tensor defined as

$$D_{ij}(\mathbf{X}, t) = \frac{\partial x_i}{\partial X_j}$$

where $x_i(\mathbf{X}, t)$ denotes the position at time t of a fluid particle that was at the position $x_i(\mathbf{X}, t_0) = X_i$ at the initial time t_0. D_{ij} thus describes the variation of the position of a particle at the current time when one slightly changes the initial position. The fluid particle obeys $dx_i/dt = u_i(\mathbf{x}, t)$, and differentiating this expression with respect to X_j, one obtains the evolution equation for D_{ij}:

$$\frac{dD_{ij}}{dt} = A_{ik} D_{kj} \quad (1)$$

The value A_{ij} is called velocity gradient tensor. Since $u_i(\mathbf{x}, t)$ is the turbulent velocity A_{ij} is a stochastic process. Many important properties of the developed turbulence such as geometric and statistical information, the alignment of vorticity with respect to the strain-rate eigenvectors, rate of deformation and shapes of fluid material volumes, non-Gaussian statistics, and intermittency, encode in the tensor A_{ik}. In the inertial range of turbulence, similar properties can be described using the coarse-grained or filtered velocity gradient tensor. Strictly speaking to get $A_{ij}(t)$ you can differentiate coarse-grained velocity only. But according to experiments [6,7] and numerical simulations [8,9], the trajectory of a Lagrange particle in the developed turbulence flow consists of two parts: regular one and trapped one into vortex filaments. But vortex filament moves also as a set of Lagrangian particles. Thus if you consider the distance between two particles which one of them in a filament and another one in a regular, smooth flow (or in another filament) you can at first approximation to consider smooth

E-mail addresses: asil72@mail.ru (A.S. Il'yn), zybin@lpi.ru (K.P. Zybin).

http://dx.doi.org/10.1016/j.physleta.2014.12.034
0375-9601/

part flow **u** only. Actually, small oscillations of one particle around vortex center do not affect on global distance growth between two particles.

Below we will discuss some general properties of the tensor D_{ij} proposing arbitrary stochastic process $A_{ik}(t)$.

3. Properties of A_{ik} in stationary uniform and isotropic turbulence

It is naturally to suppose that stationary, uniform and isotropic turbulence flow is a general stochastic process defined by probability distribution functional $P[\mathbf{u}(\mathbf{r},t)]$. The conditions of stationarity, uniformity and isotropy take the form:

$$P\big[\mathbf{u}(\mathbf{r},t)\big]=P\big[\mathbf{u}(\mathbf{r},t+t_0)\big],\qquad P\big[\mathbf{u}(\mathbf{r},t)\big]=P\big[\mathbf{u}(\mathbf{r}+\mathbf{r}_0,t)\big],$$
$$P\big[\mathbf{u}(\mathbf{r},t)\big]=P\big[R\mathbf{u}(R\mathbf{r},t)\big] \tag{2}$$

These relations must be valid for any moment t_0, position $\mathbf{r}_0$ and rotation matrix R.

Analogous one can define probability functional $P_A[A(t)]$, for any matrix $A_{ij}(\mathbf{r}_0,t)$, here $\mathbf{r}_0$ is some point fixed in space. The uniformity means that $P_A[A(t)]$ does not depend on $\mathbf{r}_0$. Any mean value of functional $F[A(t)]$ is defined by functional integral

$$\big\langle F\big[A(t)\big]\big\rangle=\int \Pi_t dA(t)\, P_A\big[A(t)\big]F\big[A(t)\big]$$

The stationarity and isotropy of the flow means that

$$P_A\big[A(t)\big]=P_A\big[A(t+t_0)\big],\qquad P_A\big[A(t)\big]=P_A\big[R^{-1}A(t)R\big]$$

Let us suppose below that stochastic process $A(t)$ is statistically independent at different moments of time:

$$P_A\big[A(t)\big]=\Pi_t p\big(A(t)\big)$$

here $p(A(t))$ some universal function of the traceless matrix $A(t)$.

It is common knowledge that any traceless matrix $A(t)$ could be split into symmetric and antisymmetric parts:

$$A=B+\Omega,\qquad B=\frac{1}{2}\big(A+A^T\big),\qquad \Omega=\frac{1}{2}\big(A-A^T\big)$$
$$B^T=B,\qquad \Omega^T=-\Omega,\qquad \Omega_{ij}=\epsilon_{ijk}\omega_k$$

here ω_k is a polar vector.

Because of SO(3) symmetry function $p(A)$ depends on rotational invariants only. For traceless matrix A, probability $p(A)$ depends on 5 invariants. Let us restrict our consideration by analytic functions $p(A)$. In this case you can choose the following invariants:

$$tr\,B^2\qquad tr\,B^3\qquad \omega^2$$
$$tr\,B\Omega^2\qquad tr\,B^2\Omega^2$$

It is important that ω in these invariants is a quadratic form only. In this case

$$P\big(A^T\big)=P(A) \tag{3}$$

4. Lyapunov exponents

For further consideration let us introduce matrix Q

$$\frac{dQ}{dt}=QA,\qquad Q(t_0)=I \tag{4}$$

here I is a unit matrix. One can see from (1) that matrix Q is connected with D by simple relation: $D=Q^T(A^T)$. The matrix Q is more convenient object, so we will discuss below its properties.

To examine the solution of (4), we proceed to a discrete approximation. Consider a discrete sequence of moments separated by Δt and let $A_{ij}(t)=(A_n)_{ij}$ be constant inside each small (n-th) interval. Then, for each Δt, the solution to Eq. (4) is described by an exponent and we get

$$Q_n=Q_{n-1}e^{A_n\Delta t}$$

The matrix Q_N is a multiplication of N random real unimodular matrices with the same distribution (discreet t-exponent). The asymptotic behavior of this object has been studied carefully and a number of important results have been obtained. (For a short summation of them, see [10].) In particular, the following theorems have been proved for reasonable conditions.

Let us consider the Iwasawa decomposition of the matrix Q_N:

$$Q=z(Q)d(Q)s(Q), \tag{5}$$

where z is an upper triangular matrix with diagonal elements equal to 1, d is a diagonal matrix with positive eigenvalues $d=\{d_1,d_2,d_3\}$, and s is an orthogonal matrix.

Theorem 1. *From [11] we have that with probability 1, there exists the limit $\lim_{N\to\infty}\frac{1}{N}d_i(Q)=\lambda_i$, where λ_i are not random, i.e., do not depend on the realization of the process $A_{ij}(t)$ but only on the statistical properties of the process, and $\lambda_1<\lambda_2<\lambda_3$, with the ordering due to the triangular matrix, which provides the inequality of the axes.*

Theorem 2. *From [12,13] we have that the distribution of $\xi_i=\frac{1}{\sqrt{N}}\times(\ln d_i(Q_N)-\lambda_i N)$ is asymptotically close to a Gaussian distribution and (weakly) converges to it as $N\to\infty$.*

Theorem 3. *From [14] we have that with probability 1, $z(Q_N)$ converges as $N\to\infty$; contrary to λ_i, the values $z_\infty=z(Q_\infty)$ are different in different realizations of $A_{ij}(t)$.*

Theorem 4. *From [15] we have that the values $\xi_i(Q_N)$ and $z(Q_N)$ are asymptotically independent.*

To calculate the Lyapunov exponents let us introduce matrix

$$\Gamma=QQ^T \tag{6}$$

according to (5),

$$\Gamma=zd^2z^T$$

Taking into account that z is upper triangle matrix one can get:

$$d_3^2=\Gamma_{33},\qquad d_1^2=\big(\big(\Gamma^{-1}\big)_{11}\big)^{-1},\qquad d_2^2=\big(\Gamma^{-1}\big)_{11}(\Gamma_{33})^{-1}$$

Basing on these relations and theorems it is easy to get Lyapunov's exponents:

$$\lambda_3=\lim_{t\to\infty}\frac{\ln\Gamma_{33}}{2t},\qquad \lambda_1=-\lim_{t\to\infty}\frac{\ln(\Gamma^{-1})_{11}}{2t},$$
$$\lambda_2=-\lambda_1-\lambda_3$$

and expressions for the Gaussian noise $\xi(t)$:

$$e^{2\int\xi_3(t)dt}=e^{-2\lambda_3 t}d_3^2=e^{-2\lambda_3 t}\Gamma_{33}$$
$$e^{-2\int\xi_1(t)dt}=e^{\lambda_1 t}d_1^2=e^{\lambda_1 t}\big(\Gamma^{-1}\big)_{11}$$

These relations are exact. It is important that these limits exist with probability 1. That is why it is possible to calculate these limits by averaging over process A with probability P_A: