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Probabilistic model of N correlated binary random variables and non-extensive statistical mechanics

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ABSTRACT

The framework of non-extensive statistical mechanics, proposed by Tsallis, has been used to describe a variety of systems. The non-extensive statistical mechanics is usually introduced in a formal way, thus simple models exhibiting some important properties described by the non-extensive statistical mechanics are useful to provide deeper physical insights. In this article we present a simple model, consisting of a one-dimensional chain of particles characterized by binary random variables, that exhibits both the extensivity of the generalized entropy with $q < 1$ and a q-Gaussian distribution in the limit of the large number of particles.

1. Introduction

There exist a number of systems featuring long-range interactions, long-range memory, and anomalous diffusion, that possess anomalous properties in view of traditional Boltzmann–Gibbs statistical mechanics. Non-extensive statistical mechanics is intended to describe such systems by generalizing the Boltzmann–Gibbs statistics [1–3]. In general, the non-extensive statistical mechanics can be applied to describe the systems that, depending on the initial conditions, are not ergodic in the entire phase space and may prefer a particular subspace which has a scale invariant geometry, a hierarchical or multifractal structure. Concepts related to the non-extensive statistical mechanics have found applications in a variety of disciplines: physics, chemistry, biology, mathematics, economics, and informatics [4–6].

The non-extensive statistical mechanics is based on a generalized entropy [1]

$$S_q = \frac{1 - \int [p(x)]^q dx}{q-1}, \tag{1}$$

where $p(x)$ is a probability density function of finding the system in the state characterized by the parameter x, while q is a parameter describing the non-extensiveness of the system. Entropy (1) is an extension of the Boltzmann–Gibbs entropy

$$S_{\mathrm{BG}} = -\int p(x) \ln p(x) dx \tag{2}$$

which is recovered from Eq. (1) in the limit $q \to 1$ [1,2]. More generalized entropies and distribution functions are introduced in Refs. [7,8]. Statistics associated to Eq. (1) has been successfully applied to phenomena with the scale-invariant geometry, like in low-dimensional dissipative and conservative maps in the dynamical systems [9–11], anomalous diffusion [12,13], turbulent flows [14], Langevin dynamics with fluctuating temperature [15,16], spin-glasses [17], plasma [18] and to the financial systems [19–21].

By maximizing the entropy (1) with the constraints $\int_{-\infty}^{+\infty} p(x)dx = 1$ and

$$\frac{\int_{-\infty}^{+\infty} x^2 [p(x)]^q dx}{\int_{-\infty}^{+\infty} [p(x)]^q dx} = \sigma_q^2, \tag{3}$$

where σ_q^2 is the generalized second-order moment [22–24], one obtains the q-Gaussian distribution density

$$p_q(x) = C \exp_q\left(-A_q x^2\right). \tag{4}$$

Here $\exp_q(\cdot)$ is the q-exponential function, defined as

$$\exp_q(x) \equiv \left[1 + (1-q)x\right]_+^{\frac{1}{1-q}}, \tag{5}$$

with $[x]_+ = x$ if $x > 0$, and $[x]_+ = 0$ otherwise. The q-Gaussian distribution (or distribution very close to it) appears in many physical systems, such as cold atoms in dissipative optical lattices [25], dusty plasma [18], motion of hydra cells [26], defect turbulence [27] and seismic activity [28]. The q-Gaussian distributions are found to be related to long-lived quasi-stationary chaotic states

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in multi-dimensional Hamiltonian systems (in Fermi–Pasta–Ulam model [29–31], in Klein–Gordon disordered lattices [32]) and in Galactic dynamics [33]. The q-Gaussian distribution is one of the most important distributions in the non-extensive statistical mechanics. Its importance stems from the generalized central limit theorems [34–36]. According to q-generalized central limit theorem, q-Gaussian can result from a sum of N q-independent random variables. The q-independence is defined in [34] through the q-product [37,38], and the q-generalized Fourier transform [34]. When $q \neq 1$, q-independence corresponds to a global correlation of the N random variables. However, the rigorous definition of q-independence is not transparent enough in physical terms.

The non-extensive statistical mechanics is introduced in a formal way, starting from the maximization of the generalized entropy [1]. Therefore, simple models providing some degree of intuition about non-extensive statistical mechanics can be useful for understanding it. There has been some effort to create such simple models. In Ref. [39] a system composed of N distinguishable particles, each particle characterized by a binary random variable, has been constructed so that the number of states with non-zero probability grows with the number of particles N not exponentially, but as a power law. For such a system in the limit $N \to \infty$ the ratio $S_q(N)/N$ is finite not for the Boltzmann–Gibbs entropy but for the generalized entropy with some specific value of q. The starting point in the construction is the Leibnitz triangle, then initial probabilities are redistributed into a small number of all the other possible states, in such a way that the norm is preserved. For example, in the restricted uniform model [39] for a fixed value of N all nonvanishing probabilities are equal. In the proposed models that yield $q \neq 1$ there are $d+1$ non-zero probabilities and the value of q is given by $q = 1 - 1/d$.

In Refs. [40,41], the goal has been to construct simple models providing q-Gaussian distributions. As in [39], the models considered in [40] consist of N independent and distinguishable binary variables, each of them having two equally probable states. The models presented in [40] are strictly scale-invariant, however, they do not approach a q-Gaussian form when the number of particles N in the model increases [42]. The situation is different with the models presented in [41]: the two proposed models do approach a q-Gaussian form, the second of them does so by construction. All models in [40,41], except the last model of [41] are for $q \leq 1$. The drawback of the models from Ref. [41] is that the standard Boltzmann–Gibbs entropy remains extensive. In addition, the models are constructed artificially and it is hard to see how they can be related to real physical systems.

The goal of this paper is to provide a simple model that achieves both the extensivity of the generalized entropy with $q \neq 1$ and q-Gaussian distribution in the limit of the large number of particles. In addition, we want to construct a model that is closer to situations in physical systems. We expect that such a model can provide deeper insights into non-extensive statistical mechanics than the previously constructed simple models.

The Letter is organized as follows: To highlight differences from our proposed model, a simple model consisting of uncorrelated binary random variables and leading to extensive Boltzmann–Gibbs entropy and a Gaussian distribution is presented in Section 2. In Section 3 we construct a simple model exhibiting the extensivity of the generalized entropy with $q \neq 1$ and q-Gaussian distribution in the limit of the large number of particles. Section 4 summarizes our findings.

2. Model of uncorrelated binary random variables

At first let us consider a model consisting of N uncorrelated binary random variables. Physical implementation of such a model could be N particles of spin $\frac{1}{2}$, the projection of each spin to the z axis can acquire the values $\pm\frac{1}{2}$. The microscopic configuration of the system can be described by a sequence of spin projections $s_1 s_2 \dots s_N$, where each $s_i = \pm\frac{1}{2}$. There are $W = 2^N$ different microscopic configurations. As is usual in statistical mechanics for the description of a microcanonical ensemble, we assign to each microscopic configuration the same probability. Thus the probability of each microscopic configuration is

$$P = \frac{1}{W} = \frac{1}{2^N}. \tag{6}$$

Note, that this system has a property of composability: if we have two spin chains with W_1 and W_2 microscopic configurations, then we can join them to form a larger system. The description of a larger system is just concatenation of the descriptions of each subsystems and the number of microscopic configurations of the whole system is $W = W_1 W_2$. The standard Boltzmann–Gibbs entropy $S_{\mathrm{BG}} = k_{\mathrm{B}} \ln W$ is extensive for this system: $S_{\mathrm{BG}} = N k_{\mathrm{B}} \ln 2$ grows linearly with N.

Let us consider a macroscopic quantity, the total spin of the system

$$M = \sum_{i=1}^{N} s_i. \tag{7}$$

The total spin can take values $M = -\frac{N}{2}, -\frac{N}{2}+1, \dots, \frac{N}{2}-1, \frac{N}{2}$. The value of M can be obtained when there are $n = M + \frac{N}{2}$ spins with the projection $+\frac{1}{2}$, the remaining spins have projection $-\frac{1}{2}$. The macroscopic configuration corresponding to the given value of M can be realized by $\binom{N}{n}$ microscopic configurations, thus using Eq. (6) the probability of each macroscopic configuration is

$$P_M = \frac{1}{2^N}\binom{N}{n}, \quad n = M + \frac{N}{2}. \tag{8}$$

Here $\binom{N}{n}$ is the binomial coefficient. Note, that the probabilities of macroscopic configurations are normalized:

$$\sum_{M=-N/2}^{N/2} P_M = 1. \tag{9}$$

Using Eq. (8) we can calculate the average spin of the system $\langle M \rangle = 0$ and the standard deviation

$$\sqrt{\langle M^2 \rangle - \langle M \rangle^2} = \frac{\sqrt{N}}{2}. \tag{10}$$

From Eq. (10) it follows that the relative width of the distribution of the total spin M decreases with number of spins N as $\frac{1}{\sqrt{N}}$. When N is large then we can approximate the factorials using Stirling formula

$$n! \approx \sqrt{2\pi n} n^n e^{-n} \tag{11}$$

and obtain a Gaussian distribution

$$P_M \approx \frac{1}{\sqrt{\pi \frac{N}{2}}} e^{-\frac{2M^2}{N}} \tag{12}$$

The Gaussian distribution can be obtained by maximizing the Boltzmann–Gibbs entropy (2) with appropriate constraints.

3. Model of correlated spins

In this section we investigate a system consisting of N correlated binary random variables. Similarly as in the previous section, we can think of a one-dimensional spin chain consisting of N spins. However, the spins are correlated: spins next to each other