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Calculating relaxation time distribution function from power spectrum based on inverse integral transformation method

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ABSTRACT

A novel method is presented for obtaining the distribution function of relaxation times $G(\tau)$ from power spectrum $1/f^{\alpha}$ $(1 \le \alpha \le 2)$. It is derived using McWhorter model and its inverse Stieltjes transform. Unlike the pre-assumed conventional $g(\tau)$ distribution, the extracted $G(\tau)$ has a peak whose width increases as the slope of the power spectrum α decreases. The peak position determines the dominant time constant of the system. Our method is unique because the distribution function is directly extracted from the measured power spectrum. We then demonstrate the validity of this method in the analysis of noise in transistor.

1. Introduction

$S(f) \sim 1/f^{\alpha}$ spectral densities, where f is the frequency and α is power component, are widely observed in nature and are of great interest in many research fields [1–9]. To explain $1/f$ spectrum, McWhorter model has been often used in a low-frequency electronic noise in the field of semiconductor research [2,5,6,9]. According to the model, $1/f^{\alpha}$ $(1 \le \alpha \le 2)$ noise can be obtained by summation of $1/f^2$ noise with the appropriate weights. In other words, the power spectrum of $S(\omega)$, as a function of angular frequency ω, can be expressed as a superposition of an uncoupled Lorentzian kernel in the following Fredholm integral equation of the first kind, namely, the suitable combination of Lorentzian kernel having different time constants τ:

$$S(\omega) = \int_{\tau_1}^{\tau_2} g(\tau) \frac{\tau}{1+\omega^2\tau^2} d\tau, \tag{1}$$

where $g(\tau)$ is the distribution function of relaxation times on the linear time scale τ, and τ_1 and τ_2 are the limits of the integration. The relationship between the distribution $g(\tau)$ and power component α has already been studied [4]. It is well known that integrating Eq. (1) with power law distribution $g(\tau) \propto 1/\tau^{(2-\alpha)}$ approximately gives power spectrum $S(\omega) \sim 1/\omega^{\alpha}$. Here, it is assumed that the $g(\tau)$ are expressed as either of relationships given below:

$$g(\tau) = \frac{a}{\tau}, \tag{2}$$

$$g(\tau) = a, \tag{3}$$

where a is a constant. Integrating Eq. (1) with Eqs. (2) and (3) gives

$$S(\omega) = \frac{a}{\omega}\left[\arctan(\omega\tau_2) - \arctan(\omega\tau_1)\right] \sim \frac{1}{\omega}, \tag{4}$$

$$S(\omega) = \frac{a}{2\omega^2} \ln\left[\frac{1+\omega^2\tau_2^2}{1+\omega^2\tau_1^2}\right] \sim \frac{1}{\omega^2}, \tag{5}$$

respectively. Note that the power spectrum $S(\omega)$ could not be always fitted well with an experimentally-obtained power spectrum $1/\omega^{\alpha}$ using the pre-assumed power law distribution $g(\tau)$ [9]. To discuss the distribution $g(\tau)$ in detail, it is absolutely required to define the distribution $g(\tau)$ directly from the experimentally-obtained power spectrum $1/\omega^{\alpha}$. Thus, further investigation is still needed to clarify the relationship between the distribution and power spectrum.

McWhorter model is Fredholm integral equation of the first kind with Lorentzian kernel. Therefore, using an appropriate spectral density, the distribution can be defined by an inverse integral

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transform. In this study, applying this alternative approach, we demonstrate that we can obtain the generalized distribution function of relaxation times from $1/f^{\alpha}$ spectrum ($1 \le \alpha \le 2$) on the basis of an inverse Stieltjes transform.

2. Generalized equation and approximate solutions of relaxation distribution

We have deduced the generalized relaxation time distribution from the viewpoint of the inverse integral transform, as described below. The basic idea of the method applied to solving the integral equation has been shown by Fujita in the field of polymer physics [10].

Owing to the wide time range covered [11], it is even more common and preferred to define the distribution in a logarithm of time instead of taking the linear time scale. Also, the interval of integration $[\tau_1, \tau_2]$ in Eq. (1) is replaced by $[-\infty, \infty]$ leading to

$$S(\omega) = \int_{-\infty}^{\infty} G(\tau)\frac{\tau}{1+\omega^2\tau^2} d\ln\tau. \tag{6}$$

The connection between both distributions is clearly given by

$$G(\tau) = \tau g(\tau). \tag{7}$$

The determination of relaxation time distribution $G(\tau)$ can be obtained with inverse Stieltjes transform of power spectrum $S(\omega)$, as detailed in Appendix A. From the theory of the inverse transformation [10,12], the second and third-order approximation of relaxation time distribution $G(\tau)$ can be given by

$$G_2(\tau) = -\frac{e^2}{2\pi\tau}\left[\frac{dS(\omega)}{d(\ln\omega)} + \frac{1}{2}\frac{d^2S(\omega)}{d(\ln\omega)^2}\right]_{\omega=1/\tau}, \tag{8a}$$

$$G_3(\tau) = -\frac{1}{\tau}\left[\frac{dS(\omega)}{d(\ln\omega)} + \frac{3}{4}\frac{d^2S(\omega)}{d(\ln\omega)^2} + \frac{1}{8}\frac{d^3S(\omega)}{d(\ln\omega)^3}\right]_{\omega=1/\tau}. \tag{8b}$$

The higher-order approximate distributions $G_n(\tau)$ can be derived in a similar and straightforward manner (see Appendix A).

When the experimentally-obtained power spectrum $S(\omega)$ is smooth enough in the wide frequency range, the distribution function $G(\tau)$ might be obtained from Eqs. (8a), (8b) by using a numerical differentiation (*e.g.*, the Savitzky–Golay method [13]). Alternatively, the distribution function $G(\tau)$ might be obtained from the power spectrum $S(\omega)$ using numerical inverse integral transform algorithms such as histogram method [14,15], CONTIN [16,17], maximum entropy method [18] and so on. In this study, to avoid the impact of quality of the experimentally-obtained power spectrum [19], we used the following equation for power spectrum $S(\omega)$ on the calculation of the distribution function $G(\tau)$

$$S(\omega) = \frac{c\tau_c}{1+(\omega\tau_c)^{\alpha}}, \tag{9}$$

where c is a constant, $\omega = 2\pi f$, $\tau_c = 1/(2\pi f_c)$, f_c is the corner frequency, and α is the slope of the power spectrum. Eq. (9) was empirically found to fit experimental data of power spectra well in the case of low-frequency noise in SiN_x insulator-gate GaAs-based etched nanowire field effect transistors (FETs) [9]. The effective time constant τ_c and the slope α were indeed successfully evaluated by using this relatively simple and flexible function. To calculate $G(\tau)$, we used the 6th-order approximate distribution (see Appendix A, Eq. (A.6c)). In general, as the order increases, the higher the order used the residual sum of squares (RSS) decreases asymptotically to a small finite value almost equal to zero. This indicates that the higher the order, the less is the error between the measured data and model. On the other hand, the calculation complexity increases as the order increases. Considering the trade-off between precision and complexity, in this work, we have chosen the 6th-order approximation because the RSS value is almost saturated around the 6th-order. Any improvement beyond the 6th-order is trivial and insignificant. The obtained $G(\tau)$ is shown in Fig. 1 for several values of noise exponent α and time constant τ_c. We used the constant $c = 1$ in all the calculations. In this plot, the $G(\tau)$ was normalized by using the trapezoidal numerical integration rule,

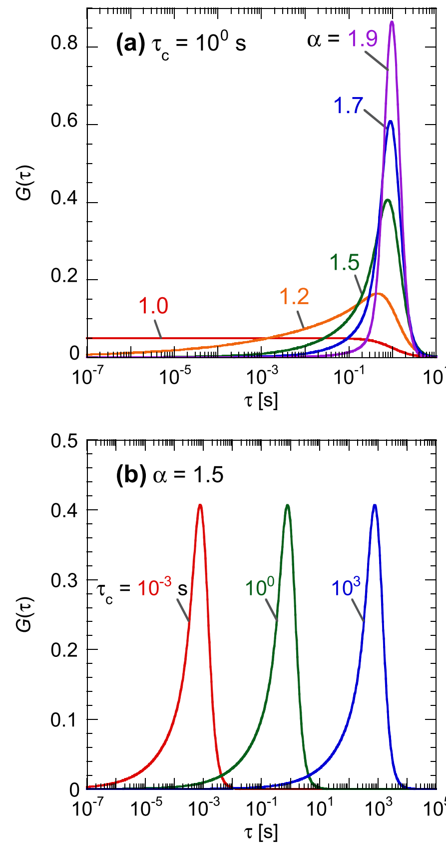

Fig. 1. Normalized distribution function $G(\tau)$ obtained by Eqs. (9), (10) and (A.6c) for (a) several values of noise exponent α and (b) time constant τ_c.

$$\int_{-\infty}^{\infty} G(\tau) d\ln\tau = 1. \tag{10}$$

As shown in Fig. 1(a), the distribution width decreases with increasing slope α. For $\alpha = 1$, it reduces to Kirton and Uren trapezium model [5,6], namely the $G(\tau)$ distribution becomes uniform-like distribution. In order to explain $1/f$ noise in semiconductor devices, they used the relationship $g(\tau)d\tau \propto d\tau/\tau = d\ln\tau$. It was therefore assumed that τ values are uniformly distributed along the $\ln\tau$ axis. When α goes to 2, the $G(\tau)$ distribution approaches delta function whose peak position corresponds to relaxation time. For intermediate values of α, the relaxation time constant of the dominant event is readily obtained as it corresponds to the peak position of asymmetrical bell-shape $G(\tau)$ distribution. Fig. 1(b) shows calculated $G(\tau)$ distribution with $\alpha = 1.5$ for different values of τ_c which shows that the $G(\tau)$ distribution retains it characteristic asymmetric bell-shape with a peak whose position corresponding to τ_c.

To verify the accuracy of the approximation and obtain back the power spectra $S(\omega)$ from the approximate $G(\tau)$, unnormalized distribution functions are substituted in the integral equation (6)