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On the inequivalence of renormalization and self-adjoint extensions for quantum singular interactions

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Abstract

A unified S-matrix framework of quantum singular interactions is presented for the comparison of self-adjoint extensions and physical renormalization. For the long-range conformal interaction the two methods are not equivalent, with renormalization acting as selector of a preferred extension and regulator of the unbounded Hamiltonian.

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1. Introduction

Singular quantum-mechanical systems involve interactions whose strongly divergent behavior at a given point governs the leading physics [1]. This entails a hierarchy in which the usual preponderance of "kinetic terms" (the Laplacian and concomitant centrifugal potentials) is suppressed by the near-singularity dominance of the interaction. In turn, the singular behavior implies an indeterminacy in the boundary condition at the singularity [1,2]. Consequently, the standard technology of regular quantum mechanics is supplemented by a mandatory regularization toolbox: either von Neumann's method of self-adjoint extensions [3–6], which addresses the boundary-value problem, or field-theory renormalization [7–12], which deals with the underlying ultraviolet cause of such indeterminacy.

A crucial question in the theory of singular potentials is whether these two apparently distinct methods are equivalent or not. In Ref. [7], their equivalence was shown for the delta-function conformal interaction. However, the more complex issue of their equivalence for the long-range conformal interaction has not been exhaustively studied. Moreover, in the absence of a systematic comparison, it is usually conjectured that both methods have comparable efficacy and ultimately yield solutions in one-to-one correspondence.

In this Letter we show that the above "equivalence conjecture" is incorrect, within the scope of traditional physical regularizations, for which the singularity emerges from a $(0+1)$-dimensional *effective* field theory [13]. Specifically, the generalization of Ref. [7] breaks down due to the spectral properties of long-range singular interactions. We analyze these issues within a comparative S-matrix framework of self-adjoint-extensions and renormalization. Furthermore, for the conformal interaction, we show that: (i) physical regularization selects a preferred self-adjoint extension for "medium-weak coupling"; (ii) the unbounded nature of the Hamiltonian for "strong coupling" can

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only be fixed by physical renormalization—otherwise, a spectrum not bounded from below would yield an unstable system.

2. S-matrix framework for singular quantum mechanics

For the complete characterization of the physics of a singular system, it proves useful to introduce a unified S-matrix framework. In addition to generating its built-in scattering observables, the S-matrix permits a comprehensive analysis of the spectrum of singular potentials, with the bound-state sector displayed through its poles. Specifically, for the family of *long-range singular interactions* $V(\mathbf{r}) = -\lambda r^{-\gamma}$ (with $\gamma \geqslant 2$), the multidimensional wave function $\Psi(\mathbf{r}) = Y_{lm}(\mathbf{\Omega})u(r)/r^{(d-1)/2}$ in $d = 2(\nu+1)$ spatial dimensions leads to

$$\left[\frac{d^2}{dr^2} + k^2 + \frac{\lambda}{r^\gamma} - \frac{(l+\nu)^2 - 1/4}{r^2}\right]u(r) = 0 \tag{1}$$

(in natural units [10]). The observables are encoded in the S-matrix through the asymptotics of two independent solutions $u_{1,2}(r)$ of Eq. (1), with

$$u(r) \propto \hat{S}u_1(r) + u_2(r), \tag{2}$$

where we will adopt the convention

$$u_{1,2}(r) \overset{(r\to\infty)}{\propto} \frac{1}{\sqrt{k}} e^{\pm ikr} e^{\mp i\pi/4} \tag{3}$$

(up to a real numerical factor) leading to the factorization

$$S = e^{i\pi(l+\nu)}\hat{S} \tag{4}$$

for the usual definition of S-matrix.

In addition, two convenient solutions $u_\pm(r)$ of Eq. (1) can be introduced as "singularity probes", to fully capture the characteristic behavior of the theory as $r \sim 0$. They can be defined in terms of the leading WKB behavior near the origin [2], which is "asymptotically exact" [14]. Correspondingly,

$$u(r) \propto \Omega u_+(r) + u_-(r), \tag{5}$$

where Ω is a "singularity parameter". At the practical level, Eqs. (2) and (5) represent two different resolutions of the wave function [15], which can be compared, provided that the basis expansions

$$u_j(r) = \alpha_j^\sigma u_\sigma(r) \tag{6}$$

(where the summation convention is adopted, with $j = 1, 2$, and $\sigma = \pm$) are established. Thus, the relation between the "components" $\hat{S}$ in Eq. (2) and Ω in Eq. (5) follows by inversion of the transfer matrix $[\alpha_j^\sigma]$. In particular, the reduced S-matrix $\hat{S} = \hat{S}(\Omega)$ is given by the fractional linear transformation

$$\hat{S} = \frac{\alpha_2^- \Omega - \alpha_2^+}{-\alpha_1^- \Omega + \alpha_1^+}, \tag{7}$$

which will play a crucial role for the remainder of the Letter.

3. Self-adjoint extensions: Conformal S-matrix

The main goal of our Letter is to highlight the failure of the "equivalence conjecture" for long-range singular interactions. As we will see, this in part due to the fact that the extensions do not describe a unique physical system but an *ensemble* of systems labeled by extension parameters. Thus, the selection of the relevant solution involves identifying the appropriate physical system within the ensemble; in short, as stated in Ref. [5], this is not a mathematical "technicality" but is to be constrained by the physics.

Let us start by stating a singular potential problem [1] within the method of self-adjoint extensions [4]. For the case of central symmetry, this reduces to Eq. (1), whose solutions may involve an ensemble of Hamiltonians rather than a single-system Hamiltonian, thereby leading to the physical indeterminacy described above. In this section, we will demonstrate the nature of this problem by solving the Schrödinger Eq. (1) for $\gamma = 2$, i.e., for the long-range conformal interaction $V(\mathbf{r}) = -\lambda/r^2$. In our approach we will subsume the results of Ref. [16] within an effective-field theory interpretation and we will give further support to our conclusions using established physical applications. Incidentally, with an appropriate interpretation, the quantum-mechanical conformal analysis *transcends non-relativistic quantum mechanics* in an effective reduced form that includes applications to the near-horizon physics and thermodynamics of black holes with generalized Schwarzschild metrics [14,17–19], and also in gauge theories [11,20].

In our analysis, we will make use of two distinctive features of the conformal interaction: (i) the existence of a critical coupling; (ii) its SO(2, 1) conformal symmetry [11,21,22]. The critical coupling $\lambda^{(*)} = (l+\nu)^2$, which separates two distinct coupling regimes [10], is associated with a qualitative change of the solutions of Eq. (1) and its multidimensional counterpart for $\gamma = 2$. In effect, the radial wave functions $u(r) = r^{1/2}Z_s(kr)$ involve a Bessel function $Z_s(z)$ of order s, whose nature changes abruptly when the parameter

$$s^2 = \lambda^{(*)} - \lambda \tag{8}$$

goes through zero. In this Letter we show that there are *no additional physical regimes* under reasonable conditions supported by a large class of realizations. However, as we will see next, this simple description is altered by the method of self-adjoint extensions, which generically gives rise to an additional transitional coupling regime. Specifically, the physical indeterminacy of von Neumann's method takes central stage for the conformal Hamiltonian with $0 < s < 1$, which admits a one-parameter family of self-adjoint extensions [16], with

$$\lambda^{(*)} - 1 < \lambda < \lambda^{(*)} \tag{9}$$

defining a subcritical *medium-weak coupling*. The existence of this additional regime for self-adjoint extensions can be easily seen by applying the standard technique [3–5], according to which the extensions are given through the eigenfunctions $u_\pm^{(\mathrm{SA})}(r)$, with eigenvalues $\pm\mu^2 i$, where μ is an arbitrary scale [23]. The corresponding deficiency subspaces, with di-