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# Emulation of the evolution of a Bose–Einstein condensate in a time-dependent harmonic trap

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**Abstract**

We emulate the ground state of a Bose–Einstein condensate in a time-dependent isotropic harmonic trap by constructing analytic simulacra of a transformed wavefunction in the regions around the origin and far from the origin. This transformed wavefunction is obtained through a pseudoconformal transformation and is a function of new spatial and temporal variables, while the simulacra are generalisations of asymptotic solutions of the nonlinear Schrödinger equation and they are matched by requiring continuity not only of the wavefunction and of its slope, but of its curvature as well. The resulting piecewise analytic simulacra coincide almost perfectly with the numerically obtained solutions of the time-dependent nonlinear Schrödinger equation and constitute an easy and accurate analytic method for describing fully the condensate ground state.



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The discovery of Bose–Einstein condensates has created an increased interest in the nonlinear Schrödinger equation. Indeed, the Gross–Pitaevskii (GP) equation [1], a highly successful mean field approximation that yields the macroscopic wavefunction for the gaseous Bose–Einstein condensates, is just a cubic nonlinear Schrödinger equation in a trapping potential. This trapping potential term has opened up new theoretical and mathematical investigations. Most of these investigations have focussed on harmonic traps, which do not give exact analytic solutions, and have therefore consisted of numerical studies, involving elaborate computations [2]. The difficulty of the problem is compounded when the trap acquires an arbitrary time-dependence.

The evolution of the condensate in a time-dependent trap has been addressed in many different ways. There are works based on scaling arguments and the Thomas–Fermi approximation [3], numerical solutions that focus usually on watching the time evolution of a suitable initial condition [4], as well as variational methods [5]. These latter treatments give the correct qualitative behavior of the condensates, but not necessarily good quantitative agreement with the results of the numerical calculations. It would be desirable though, in many instances, to be able to obtain accurate, though approximate, analytic solutions for Bose–Einstein condensates. Such solutions would provide a simple quantitative tool for the analysis of experimental data for trapped condensed gases.

There is, indeed, a method for finding excellent analytic approximate solutions to the nonlinear Schrödinger equation in a time-independent harmonic trapping potential [6]. This method leads to simple analytic expressions and is most easily used when finding the condensate ground state. It can be generalised though to higher states and to various potentials. Its basic strategy is to construct *piecewise* analytic *simulacra* of the solutions of the GP equation. In the case of the ground state, for example, one will be looking for analytic simulacra of the solutions of this equation in two regions: one around the origin and one far from the origin. These simulacra will be constructed as generalisations of the asymptotic solutions of the GP equation in the

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two regions mentioned. The two simulacra will then be joined, requiring that the wavefunction and its slope be continuous at the junction, and that the curvatures of the joined pieces also equal each other there. Indeed, if the simulacrum is a very good one, then its curvature must be continuous everywhere, since this is the case for the exact ground state.

The solution obtained by this emulation for the ground state in a time-independent harmonic trap is indeed an excellent approximation to the solution obtained numerically, for any value of the number of atoms, large or small [6]. The simple analytic expressions for the wavefunction are very useful, because they provide a simple quantitative tool for the analysis of experimental data for trapped condensed gases, without relying on complex and extensive calculations.

In this Letter we shall generalize the work of Ref. [6] to the case of a harmonic trap that has an arbitrary time-dependence. We shall find that the analytic simulacra give again the correct behaviour with great accuracy, even though the analytic expressions used are very simple.

Let us begin with the Gross–Pitaevskii equation in a time-dependent isotropic harmonic trap:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + \frac{1}{2}m\omega^2(t)r^2\Psi + g_0 g(t)|\Psi|^{2n}\Psi = i\hbar\frac{\partial\Psi}{\partial t}, \tag{1}$$

where $g_0$ and $g(t)$ are positive (repulsive two-body interactions) with $g(0)=1$. For the sake of generality, we have adopted a time-dependent nonlinearity [7]. The variation of the nonlinearity can be achieved if we modulate the atomic scattering length by the Feshbach resonance technique. We note that the typical spatial scale $\sqrt{\hbar/2m\omega}$ for the ground state of the trap is of the order of a few microns, with $\omega$ being around 50 $s^{-1}$. The corresponding scattering length $a = mg_0/4\pi\hbar^2$ for $^{87}Rb$ is 5.77 nm, while it is 2.75 nm for the case of $^{23}Na$. There are typically a few thousand atoms in the condensate. As a result of the repulsive two-body forces, the size of the condensate can be substantially larger than the scale $\sqrt{\hbar/2m\omega}$. For example, in the case of 80000 $^{23}Na$ atoms the size of the condensate is around 20 microns [8].

We shall define the dimensionless quantities $\Omega(\tau) = \omega(t)/\omega(0)$ (with $\Omega(0)=1$), $\mathbf{x} = \mathbf{r}\sqrt{2m\omega(0)/\hbar}$, $\tau = \omega(0)t$ and $\phi(\mathbf{x},\tau) = \Psi(\mathbf{r},t)(\hbar\omega(0)/g_0)^{-1/(2n)}/\Psi_0$, where $\Psi_0$ is chosen so as to make $\phi(0,0)=1$. Then Eq. (1) reduces to the dimensionless equation

$$-\nabla_x^2\phi + \frac{1}{4}\Omega^2(\tau)x^2\phi + g(\tau)\Psi_0^{2n}|\phi|^{2n}\phi = i\frac{\partial\phi}{\partial\tau}, \tag{2}$$

when the number of the atoms in the condensed state in $d$ dimensions is proportional to the quantity $N = \int d^d\mathbf{x}\,|\phi|^2$. Let us now perform a pseudoconformal transformation (also called lens transformation) on the wavefunction [9]:

$$\mathbf{x}' = \mathbf{x}/\ell(\tau), \tag{3}$$

$$\tau' = \int_0^\tau \frac{1}{\ell^2(\eta)}\,d\eta, \tag{4}$$

$$\gamma(\tau) = \frac{1}{4\ell(\tau)}\frac{d\ell}{d\tau}, \tag{5}$$

$$\phi(\mathbf{x},\tau) = \ell^{-d/2}(\tau)\chi(\mathbf{x}',\tau')e^{i\gamma(\tau)x^2}, \tag{6}$$

where $\ell(\tau)$ is an arbitrary function that satisfies the initial condition $\ell(0)=1$ and $d$ is the number of dimensions. Note also that the transformed wavefunction $\chi$ satisfies the initial condition $\chi(0,0)=1$.

We can rewrite the dimensionless GP Eq. (2) in terms of the new spatial and temporal variables $\mathbf{x}'$ and $\tau'$:

$$-\nabla_{x'}^2\chi + \frac{1}{4}\Omega_{\text{eff}}^2 x'^2\chi + g_{\text{eff}}\Psi_0^{2n}|\chi|^{2n}\chi = i\frac{\partial\chi}{\partial\tau'}, \tag{7}$$

where

$$\Omega_{\text{eff}}^2 = \Omega^2(\tau)\ell^4(\tau) + \ell^3(\tau)\frac{d^2\ell}{d\tau^2} \tag{8}$$

and

$$g_{\text{eff}} = g(\tau)\ell^{2-nd}(\tau). \tag{9}$$

Thus the transformed wavefunction $\chi(\mathbf{x}',\tau')$ obeys a GP equation for a trap with an effective frequency $\Omega_{\text{eff}}$ and an effective nonlinearity $g_{\text{eff}}$.

This very interesting transformation property has been used before in order to describe the coherent evolution of the condensate [10] and, in particular, to provide the generalization of the Thomas–Fermi approximation for condensates in time-dependent traps [11]. It shows that, for the case $nd=2$ and $g(\tau)=1$, we can choose the arbitrary scaling function $\ell(\tau)$ so as to make $\Omega_{\text{eff}}=1$, in which case the problem in the $\mathbf{x}'$ and $\tau'$ variables reduces to the solution of the equation for a static parabolic potential of frequency 1. Thus, in this case, a complete description for the space–time evolution at an arbitrary variation of $\Omega(\tau)$ can be achieved, provided we have the solution of the static problem [12].

The transformed equation, Eq. (7), can also be used in a different way. Instead of selecting the scaling function $\ell(\tau)$ so as to make $\Omega_{\text{eff}}=1$, we could select it so as to make $g_{\text{eff}}=1$, thus reducing a problem with a time-dependent nonlinearity into a problem with a constant nonlinearity. We shall not examine this possibility any further, though, and we shall assume that the nonlinearity $g(\tau)$ is equal to 1 from now on.

This transformed Eq. (7), which has been used not only for Bose–Einstein condensates but in other areas of physics as well [13], will yield analytic expressions then, provided we can find analytic expressions for the static problem. This is feasible in the context of the Thomas–Fermi approximation, where we assume that the kinetic energy is negligible and we thus drop the $\nabla^2$ term, or in variational treatments, which give a good qualitative description, though not quantitatively reliable. It is also feasible if we emulate the ground state of the condensate, in which case the results agree perfectly with the numerical solutions [6]. We shall combine therefore the emulation of the ground state for a static harmonic potential with the transformed GP Eq. (7), in order to obtain simple and accurate analytic expressions for the ground state wavefunction of the evolving condensate.

Let us then examine separately the amplitude and the phase of $\chi(\mathbf{x}',\tau')$:

$$\chi(\mathbf{x}',\tau') = R(\mathbf{x}',\tau')e^{i\theta(\mathbf{x}',\tau')}. \tag{10}$$