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Nonadditive entropy for random quantum spin-S chains

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ABSTRACT

We investigate the scaling of Tsallis entropy in disordered quantum spin-S chains. We show that an extensive scaling occurs for specific values of the entropic index. Those values depend only on the magnitude S of the spins, being directly related with the effective central charge associated with the model.

1. Introduction

Correlations among parts of a quantum system are behind remarkable phenomena, such as a quantum phase transition (QPT) [1,2]. In particular, the relationship between correlations and QPTs is revealed by the behavior of entanglement at criticality as measured, e.g., by the von Neumann entropy (see, for instance, Ref. [3]). Given a quantum system in a pure state $|\psi\rangle$ and a bipartition of the system into two subsystems A and B, the von Neumann entropy between A and B reads

$$\mathcal{S} = -\operatorname{Tr}(\rho_A \ln \rho_A) = -\operatorname{Tr}(\rho_B \ln \rho_B), \tag{1}$$

where $\rho_A = \operatorname{Tr}_B \rho$ and $\rho_B = \operatorname{Tr}_A \rho$ denote the reduced density matrices of A and B, respectively, with $\rho = |\psi\rangle\langle\psi|$. If A and B are probabilistic independent (such that $\rho = \rho_A \otimes \rho_B$), the von Neumann entropy is additive, i.e., $\mathcal{S}_{AB} = \mathcal{S}_A + \mathcal{S}_B$. As a consequence, $\mathcal{S}$ is extensive for uncorrelated subsystems, namely, $\mathcal{S}(L) \propto L$, where L denotes the size of a block of the system. On the other hand, $\mathcal{S}$ becomes nonextensive in presence of correlations. Indeed, for critical systems in one dimension, which are known to be highly entangled, conformal invariance implies a diverging logarithmic scaling given by $\mathcal{S}(L) \propto (c/3)\ln L$ (or, more specifically, $\mathcal{S}(L) = (c/3)\ln L + \text{constant}$), where c is the central charge associated with the Virasoro algebra of the underlying conformal field theory [4–6]. For noncritical (gapful) systems in one dimension, entanglement saturates at a constant value k, i.e., $\mathcal{S}(L) \to k$ as $L \to \infty$. More generally, for higher dimensions, noncritical systems are expected to obey the area law, which implies that the von Neumann entropy of a region scales as the surface area of the region instead of the volume of the region itself. In other words, the area law establishes that $\mathcal{S}(L) \propto L^{D-1}$ ($L \to \infty$), where D is the dimension of the system.

Remarkably, it has recently been shown in Refs. [7,8] that a quantum system may exhibit specific probability correlations among its parts such that an extensive entropy can be achieved even for highly correlated subsystems. This has been obtained by generalizing the von Neumann entropy into the nonadditive Tsallis q-entropy [9,10]

$$\mathcal{S}_q[\rho] = \frac{1}{1-q}\left(\operatorname{Tr}\rho^q - 1\right), \tag{2}$$

with $q \in \mathbb{R}$. One can show that the von Neumann entropy is a particular case of Eq. (2) by taking $q = 1$. Tsallis entropy has been successfully applied to handle a variety of physical systems, in particular those exhibiting long-range interactions. Recent experimental results for its predictions can be found, e.g., in Refs. [11, 12]. In Tsallis statistics, the additivity of the von Neumann entropy for independent subsystems is replaced by the pseudo-additivity relation of the $\mathcal{S}_q$ entropy

$$\mathcal{S}_q[\rho_A \otimes \rho_B] = \mathcal{S}_q[\rho_A] + \mathcal{S}_q[\rho_B] + (1-q)\mathcal{S}_q[\rho_A]\mathcal{S}_q[\rho_B]. \tag{3}$$

The investigation of $\mathcal{S}_q$ in conformal invariant quantum systems has revealed that the extensivity of the entropy can be achieved for a particular choice q_{ext} of the entropic index q in Eq. (2). In particular, q_{ext} is directly associated with the central charge c. More specifically, the extensivity of $\mathcal{S}_q$ occurs for [8]

$$q_{ext} = \frac{\sqrt{9+c^2}-3}{c}. \tag{4}$$

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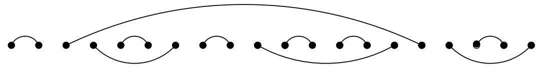

Fig. 1. A schematic picture of the RSP. Spin singlets are composed randomly at arbitrary distances.

The aim of this work is to consider the scaling of the nonadditive entropy $\mathcal{S}_q$ and, consequently, its extensivity in quantum critical spin chains under the effect of disorder into the exchange couplings among the spins. Indeed, disorder appears as an essential feature in a number of condensed matter systems, motivating a great deal of theoretical and experimental research (see, e.g., Refs. [13,14]). In particular, it is well known that, in the case of a spin-S random exchange Heisenberg antiferromagnetic chain (REHAC), disorder can drive the system to the so-called random singlet phase (RSP), which is a gapless phase described by spin singlets distributed over arbitrary distances [15]. In recent years, it has been observed that the entanglement entropy in critical random spin chains displays a logarithmic scaling that closely resembles the behavior of pure (non-disordered) systems. Indeed, for a block of spins of length L, we have that the von Neumann entropy reads $\mathcal{S}(L) \propto (c_{eff}/3)\ln L$, where c_{eff} is an effective central charge that governs the scale of the entropy [16]. Moreover, it has been shown that in the case of the RSP, c_{eff} is determined solely in terms of the magnitude S of the spin in the chain [17–19] (see Ref. [20] for a review of entanglement in random systems and Ref. [21] for other connected results). Here, we will show that the extensivity of $\mathcal{S}_q$ can also be obtained for random critical spin chains, with q_{ext} governed by c_{eff}. Hence, q_{ext} will be given as a unique function of the spin S. Moreover, as we will see, around the extensivity point q_{ext}, $\mathcal{S}_q(L) \propto L^{\gamma}$, with the exponent γ of the power law given by a quadratic function of q.

2. Nonadditive entropy for a set of random singlets

We begin by considering the typical arrange of a quantum spin-S chain in the RSP, which is provided by a set of spin singlets distributed over arbitrary distances, as sketched by Fig. 1.

In order to evaluate $\mathcal{S}_q$ in the RSP, we begin by considering a number n of singlets connecting a contiguous block composed by L spins with the rest of the chain. In this situation, the pseudo-additivity of $\mathcal{S}_q$ implies that Tsallis entropy is given by the Proposition below.

Proposition 1. *For a bipartite system composed of a number n of spin-S singlets connecting two blocks, with $n \in \mathbb{N}$, Tsallis entropy $\mathcal{S}_q^{(n)}$ for each block is given by*

$$\mathcal{S}_q^{(n)} = \frac{1}{1-q}\left[\left((2S+1)^{n(1-q)} - 1\right)\right]. \tag{5}$$

Proof. The proof can be obtained by finite induction. Indeed, the single-site reduced density operator ρ_A for a spin-S singlet can be represented by a D-dimensional diagonal matrix given by $\rho_A = \mathrm{diag}(D^{-1}, D^{-1}, \ldots, D^{-1})$, with $D = 2S+1$. Therefore, from Eq. (2), we obtain that $\mathcal{S}_q^{(1)} = (1-q)^{-1}(D^{1-q} - 1)$. For two singlets, the pseudo-additivity of $\mathcal{S}_q$ given by Eq. (3) implies that $\mathcal{S}_q^{(2)} = (1-q)^{-1}(D^{2(1-q)} - 1)$. By taking the general expression for the entropy for n singlets as $\mathcal{S}_q^{(n)} = (1-q)^{-1}(D^{n(1-q)} - 1)$, we obtain for $(n+1)$ singlets that $\mathcal{S}_q^{(n+1)} = (1-q)^{-1}(D^{(n+1)(1-q)} - 1)$. Hence, Eq. (5) holds for any $n \in \mathbb{N}$. $\square$

Tsallis entropy for the RSP can then be obtained by numerically averaging $\mathcal{S}_q^{(n)}$ over a sample of random couplings along the chain.

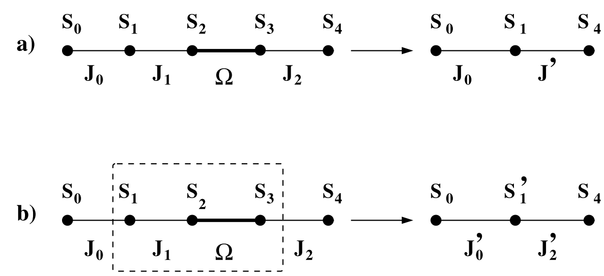

Fig. 2. Modified MDH renormalization procedure for spin-S chains.

These random configurations are generated by following a gapless probability distribution, which drives the system to the RSP, with the entropy of each configuration computed by counting the spin singlets via a renormalization group approach described in the next section.

3. Renormalization group method for random spin systems

The RSP can be conveniently handled via a perturbative real-space renormalization group method introduced by Ma, Dasgupta and Hu (MDH) [22,23], which was successfully applied to the spin-1/2 REHAC. This approach was proven to be asymptotically exact, which allowed for a fully characterization of the properties of the RSP [15]. Considering a set of random Heisenberg antiferromagnetic interactions J_i between neighboring spins S_i and S_{i+1}, the original MDH method consists in finding the strongest interaction Ω between a pair of spins (S_2 and S_3 in Fig. 2a) and treating the couplings of this pair with its neighbors (J_1 and J_2 in Fig. 2a) as a perturbation. Diagonalization of the strongest bond leads at zeroth order in perturbation theory to a singlet state between the spins coupled by Ω. Then, the singlet is decimated away and an effective interaction J' is perturbatively evaluated. By iteratively applying this procedure, the low-energy behavior of the ground state will be given by a collection of singlet pairs and the structure of the RSP will naturally appear.

Unfortunately, when generalized to higher spins, this method, at least in its simplest version, revealed to be ineffective. The reason is that, after the elimination procedure of the strongest bond Ω, the effective interaction J' may be greater than Ω. Then, the problem becomes essentially non-perturbative for arbitrary distributions of exchange interactions. For instance, considering the REHAC with arbitrary spin-S, the renormalized coupling is given by the recursive relation [24]

$$J' = \frac{2}{3}S(S+1)\frac{J_1 J_2}{\Omega}. \tag{6}$$

Notice that, for $S \geqslant 1$, the renormalization factor is $(2/3)S \times (S+1) > 1$, resulting in the breakdown of perturbation theory. In order to solve this problem, a generalization of the MDH method was proposed in Refs. [25,26] (for other proposals, see also [27,28]). This generalized MDH method consists in either of the following procedures shown in Fig. 2. Taking the case of the Heisenberg chain as an example, if the largest neighboring interaction to Ω, say J_1, is $J_1 < 3\Omega/[2S(S+1)]$, then we eliminate the strongest coupled pair obtaining an effective interaction between the neighbors to this pair which is given by Eq. (6) (see Fig. 2a). This new effective interaction is always smaller than those eliminated. Now suppose $J_1 > J_2$ and $J_1 > 3\Omega/[2S(S+1)]$. In this case, we consider the *trio* of spins-S coupled by the two strongest interactions of the trio, J_1 and Ω and solve it exactly (see Fig. 2b). This trio of spins is then substituted by one effective spin interacting with its neighbors through new renormalized interactions obtained by degenerate perturbation theory for the ground state