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Differential transform method for solving solitary wave with discontinuity ☆

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ABSTRACT

In this Letter, the differential transform method is developed to solve solitary waves governed by Camassa–Holm equation. Purely analytic solutions are given for solitons with and without continuity at crest. A Padé technique is also combined with DTM. This provides us a new analytic approach to solve soliton with discontinuity.

1. Introduction

It is difficult to solve nonlinear problems, especially by analytic techniques. In 1986, Zhou [1] first introduced the differential transform method (DTM) in solving linear and nonlinear initial value problems in the electrical circuit analysis. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. Ravi Kanth and Aruna have developed this method for PDEs and obtained closed form series solutions for both linear and nonlinear problems [2,3]. Besides the differential transform method is independent on whether or not there exist small parameters in the considered equation. Therefore, the differential transform method can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyze strongly nonlinear problems. This method has been successfully applied to solve many types of nonlinear problems [4–9]. All of the previous applications of the differential transform method deal with solutions without discontinuity. However, many nonlinear problems have different types of discontinuity. In this Letter, in order to verify the validity of the differential transform method for nonlinear problems with discontinuation, we further apply it to solve shallow solitary water wave problems governed by Camassa–Holm equation.

In the study of shallow water waves, Camassa and Holm [10] used the Hamiltonian method to derive a completely integrable wave equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

where k is a constant related to the critical shallow water wave speed, c is the phase speed, $u = u(x,t)$ denotes the velocity, x and t denote the spatial and temporal variables, respectively. Since the birth of the Camassa–Holm equation (1), a huge amount of work has been carried out to study dynamic properties of Eq. (1) [11–15].

For $k = 0$, Eq. (1) has traveling wave solutions of the form $ce^{-|x-ct|}$, called peakons, which capture an essential feature of the traveling waves of largest amplitude (see Constantin [16], Constantin and Escher [17,18] and Toland [19]). For $k > 0$, it's solitary waves are stable solitons (see Constantin and Strauss [20,21] and Johnson [22]). This is rather interesting. We employ the differential transform method to solve this solitary wave problem so as to provide a new analytic approach for nonlinear problems with discontinuity.

2. Differential transform method

2.1. Basic idea

The basic definitions and fundamental operations of the differential transform method are defined as follows [23,24]: Consider

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Table 1
The operation for differential transformation method.

Original function	Transformed function
$w(x) = g(x) + h(x)$	$W(k) = G(k) + H(k)$
$w(x) = \alpha g(x)$	$W(k) = \alpha G(k)$
$w(x) = \frac{\partial g(x)}{\partial x}$	$W(k) = (k+1)G(k+1)$
$w(x) = g(x)h(x)$	$W(k) = \sum_{r=0}^{k} G(r)H(k-r)$
$w(x) = \frac{\partial^m g(x)}{\partial x^m}$	$W(k) = (k+1)(k+2)\cdots(k+m)G(k+m)$

a function of variable $w(x)$, be analytic in the domain Ω and let $x = x_0$ in this domain. The function $w(x)$ is then represented by one series whose center located at x_0. The differential transform of the function $w(x)$ is in the form

$$W(k) = \frac{1}{k!}\left[\frac{d^k w(x)}{dx^k}\right]\bigg|_{x=x_0}, \tag{2}$$

where $w(x)$ is the original function and $W(k)$ is the transformed function.

The differential inverse transform of $U_n(k)$ is defined as

$$w(x) = \sum_{k=0}^{\infty} W(k)(x-x_0)^k. \tag{3}$$

In a real application, and when x_0 is taken as 0, then the function $w(x)$ can be expressed by a finite series and with the aid of Eq. (3), $w(x)$ can be written as

$$w(x) = \sum_{k=0}^{\infty} W(k)x^k = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k w(x)}{dx^k}\right]\bigg|_{x=0} x^k. \tag{4}$$

The Mth-order approximation of the object function $w(x)$ is given by

$$w(x) = \sum_{k=0}^{M} \frac{1}{k!}\left[\frac{d^k w(x)}{dx^k}\right]\bigg|_{x=0} x^k = \sum_{k=0}^{M} W(k)x^k. \tag{5}$$

The fundamental mathematical operations performed by generalized differential transform method are listed in Table 1.

2.2. Differential transform-Padé technique

The accuracy and convergence of the solution given by series Eq. (5) can be further enhanced by the differential transform-Padé technique. The basic idea of summation theory is to represent $f(x)$, the function in question, by a convergent expression. In Euler summation this expression is the limit of the convergent series, while in Borel summation this expression is the limit of a convergent integral. The difficulty with Euler and Borel summation is that all of the terms of the divergent series must be known exactly before the sum can be found even approximately. But in real computation, only a few terms of a series can be calculated before a state of exhaustion is reached. Therefore, a summation algorithm is needed which requires as input only a finite number of terms of divergent series. Then as each new term is given, we can give a new and improved estimate of exact sum of the divergent series. Padé approximation is a well-known summation method which having this property.

As a method of enhancing accuracy and convergence of the series, Padé approximation is widely applied [25–27]. The idea of Padé summation is to replace a power series

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n \tag{6}$$

by a sequence of rational functions which is a ratio of two polynomials

$$f_M^N(x) = \frac{\sum_{k=0}^{N} a_k x^k}{\sum_{k=0}^{M} b_k x^k}, \tag{7}$$

where we choose $b_0 = 1$ without loss of generality. We choose the remaining $(M+N+1)$ coefficients $a_0, a_1, \ldots, a_N, b_1, b_2, \ldots, b_M$, so that the first $(M+N+1)$ terms in the power series expansion of $f_M^N(x)$ match the first $(M+N+1)$ terms of the power series $f(x) = \sum_{n=0}^{+\infty} c_n x^n$. The resulting rational function $f_M^N(x)$ is called a Padé approximate. We will see that constructing $f_M^N(x)$ is very useful. If $\sum c_n x^n$ is a power series representation of the function $f(x)$, then in many instances $f_M^N(x) \to f(x)$ as $N, M \to \infty$, even if $\sum c_n x^n$ is a divergent series. Usually we consider only the convergence of the Padé sequences $f_0^J, f_1^{1+J}, f_2^{2+J}, \ldots$ having $N = M + J$ with J fixed and $M \to \infty$. If $j = 0$ then this sequence is called diagonal sequence.

It often works quite well, even beyond their proven range of applicability. We combine the differential transform with Padé technique, and call this method differential transform-Padé approximation.

3. Mathematical formulation

Under the definition $\xi = lx + ct$

$$(c + 2lk)u' + 3luu' - cl^2u''' = 2l^3u'u'' + l^3uu''', \tag{8}$$

where the prime denotes the derivative with respect to ξ.

Camassa and Holm [10] pointed out that the solitary wave solution exists when $0 \leqslant k < \frac{1}{2}$. Due to the symmetry of the solitons, we consider the wave profile only for $\xi \geqslant 0$. For simplicity, we choose $c = 1$ and $l = 1$.

3.1. Solutions with discontinuity of derivative at crest

Let us consider the first case that the first derivative at crest of the solitary waves has not continuity, corresponding to $k = 0$. In this special case, Eq. (8) reads

$$cu' + 3luu' - cl^2u''' = 2l^3u'u'' + l^3uu'''. \tag{9}$$

The corresponding boundary conditions are

$$u(0) = 1, \qquad u(+\infty) = 0. \tag{10}$$

It should be emphasized that the boundary condition $u'(0) = 0$ is invalid now. However, the derivative at crest of the solitary waves from right-hand side exists, namely, $u'_+(0), u''_+(0), \ldots, u_+^{(n)}(0)$ exist. Camassa and Holm pointed out [10] that the solitary waves with discontinuity at crest exist in the case of $k = 0$, and the corresponding exact solution is

$$u(\xi) = e^{-|\xi|}. \tag{11}$$

The transformed version of Eq. (9) is in the following form

$$\begin{aligned}
&(j+1)U(j+1) + 3\sum_{i=0}^{j}(j-i+1)U(i)U(j-i+1) \\
&\quad - (j+3)(j+2)(j+1)U(j+3) \\
&= 2\sum_{i=0}^{j}(j-i+2)(j-i+1)(i+1)U(i+1)U(j-i+2) \\
&\quad + \sum_{i=0}^{j}(j-i+3)(j-i+2)(j-i+1)U(i)U(j-i+3).
\end{aligned} \tag{12}$$