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Wavelets method for the time fractional diffusion-wave equation

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ABSTRACT

In this paper, an efficient and accurate computational method based on the Legendre wavelets (LWs) is proposed for solving the time fractional diffusion-wave equation (FDWE). To this end, a new fractional operational matrix (FOM) of integration for the LWs is derived. The LWs and their FOM of integration are used to transform the problem under consideration into a linear system of algebraic equations, which can be simply solved to achieve the solution of the problem. The proposed method is very convenient for solving such problems, since the initial and boundary conditions are taken into account automatically.

1. Introduction

Many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by models using mathematical tools from fractional calculus [1]. It is worth nothing that analytic solutions of most fractional differential equations cannot be obtained explicitly, so proposing new methods to find numerical solutions of these equations has practical importance. Due to this fact, in recent years several numerical methods were proposed for fractional differential equations, for instance see [2–11]. An important class of fractional differential equations which has been studied widely in recent years is the time fractional diffusion-wave equation (FDWE). The time FDWE is obtained from the classical diffusion-wave equation by replacing the second-order time derivative term by a fractional derivative of order $1 < \alpha \leq 2$ [12]. Many of the universal electromagnetic, acoustic, mechanical responses can be described accurately by the FDWE [16,17]. It is also worth noting that fractional diffusion equation and diffusion wave equation have a lot in common. For example, they can behave like diffusion. For more details about some important properties of the fractional diffusion equation and diffusion wave equation, the interested reader is advised to see [13–15]. In the last few years, several numerical methods have been proposed for solving FDWE, for instance see [12,16–30]. In recent years, the LWs have been applied for solving some fractional differential equations, for instance see [31–33].

In this paper, we propose an efficient and accurate computational method based on the LWs for solving the FDWE with damping [27]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t),$$
$$(x,t) \in [0,1] \times [0,1],\ 1 < \alpha \leq 2, \tag{1}$$

subject to the initial and boundary conditions:

$$\begin{cases} u(x,0) = f_0(x), & \dfrac{\partial u(x,0)}{\partial t} = f_1(x), \quad x \in [0,1], \\ u(0,t) = g_0(t), & u(1,t) = g_1(t), \quad t \in [0,1], \end{cases} \tag{2}$$

where the parameter α denotes the order of the fractional derivative in the Caputo sense, which will be described in the next section, f_0, f_1, g_0 and g_1 are given functions in $L^2[0,1]$, and q is a given function in $L^2([0,1] \times [0,1])$.

In the case $\alpha = 2$, this equation is the telegraph equation, which governs electrical transmission in a telegraph cable [28]. This equation can also be characterized as a wave equation, governing wave motion in a string, with a damping effect due to the term $\frac{\partial u(x,t)}{\partial t}$.

In order to compute the approximate solution of this equation, we first present some useful properties of the LWs and then derive a new FOM of integration for these basis functions. A collocation

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method based on hat functions (HFs) is employed to derive a general procedure for forming this matrix. The proposed method is based on reducing the problem under consideration into a linear system of algebraic equations by expanding the solution as LWs with unknown coefficients and using the FOM of integration, which can be simply solved to achieve the solution of the problem. The proposed method is very convenient for solving such problems, since the initial and boundary conditions are taken into account automatically.

The current paper is organized as follows: In Section 2, some necessary definitions and mathematical preliminaries of the fractional calculus are reviewed. In Section 3, the LWs and some of their properties are investigated. In Section 4, the proposed computational method is described for solving the FDWE (1). In Section 5, the proposed method is applied for solving some numerical examples. Finally a conclusion is drawn in Section 6.

2. Preliminaries and notations

In this section, we give some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results.

Definition 2.1. A real function $u(t)$, $t>0$, is said to be in the space C_μ, $\mu\in\mathbb{R}$ if there exists a real number p $(>\mu)$ such that $u(t)=t^pu_1(t)$, where $u_1(t)\in C[0,\infty]$ and it is said to be in the space C_μ^n if $u^{(n)}\in C_\mu$, $n\in\mathbb{N}$.

Definition 2.2. The Riemann–Liouville fractional integration operator of order $\alpha\geq 0$ of a function $u\in C_\mu$, $\mu\geq -1$, is defined as [34]:

$$\left(I^\alpha u\right)(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}u(\tau)d\tau, & \alpha>0,\\ u(t), & \alpha=0.\end{cases} \tag{3}$$

The Riemann–Liouville fractional integration operator has the following properties:

$$\left(I^\alpha I^\beta u\right)(t)=\left(I^{\alpha+\beta}u\right)(t), \qquad I^\alpha t^\vartheta=\frac{\Gamma(\vartheta+1)}{\Gamma(\alpha+\vartheta+1)}t^{\alpha+\vartheta}, \tag{4}$$

where $\alpha,\beta\geq 0$ and $\vartheta>-1$.

Definition 2.3. The fractional derivative operator of order $\alpha>0$ in the Caputo sense is defined as [34]:

$$\left(D_*^\alpha u\right)(t)=\begin{cases}\frac{d^nu(t)}{dt^n}, & \alpha=n\in\mathbb{N},\\ \frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau, & n-1<\alpha<n,\end{cases} \tag{5}$$

where n is an integer, $t>0$, and $u\in C_1^n$.

The useful relation between the Riemann–Liouville operator and Caputo operator is given by the following expression [34]:

$$\left(I^\alpha D_*^\alpha u\right)(t)=u(t)-\sum_{k=0}^{n-1}u^{(k)}\left(0^+\right)\frac{t^k}{k!},\quad t>0,\ n-1<\alpha\leq n, \tag{6}$$

where n is an integer, and $u\in C_1^n$.

3. The LWs and their properties

The LWs $\psi_{nm}(t)=\psi(k,n,m,t)$ have four arguments, $n=1,2,\ldots,2^k$, k is any arbitrary non-negative integer, m is the degree of the Legendre polynomials and independent variable t is defined on $[0,1]$. They are defined on the interval $[0,1]$ by [32]:

$$\psi_{nm}(t)=\begin{cases}\sqrt{2m+1}2^{\frac{k}{2}}P_m(2^{k+1}t-2n+1), & t\in[\frac{n-1}{2^k},\frac{n}{2^k}],\\ 0, & \text{o.w.}\end{cases} \tag{7}$$

Here, $P_m(t)$ are the well-known Legendre polynomials of degree m, which are orthogonal with respect to the wight function $w(t)=1$, on the interval $[-1,1]$ [35]. The set of the LWs is an orthogonal set with respect to the weight function $w(t)=1$.

A function $u(t)$ defined over $[0,1]$ may be expanded by the LWs as:

$$u(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(t), \tag{8}$$

where $c_{nm}=(u(t),\psi_{nm}(t))$ and $(.,.)$ denotes the inner product in $L^2[0,1]$.

By truncating the infinite series in (8), we can approximate $u(t)$ as follows:

$$u(t)\simeq\sum_{n=1}^{2^k}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(t)=C^T\Psi(t), \tag{9}$$

where T indicates transposition, C and $\Psi(t)$ are $\hat{m}=2^kM$ column vectors.

For simplicity, Eq. (9) can be also written as:

$$u(t)\simeq\sum_{i=1}^{\hat{m}}c_i\psi_i(t)=C^T\Psi(t), \tag{10}$$

where $c_i=c_{nm}$ and $\psi_i(t)=\psi_{nm}(t)$, and the index i is determined by the relation $i=M(n-1)+m+1$.

Thus we have:

$$C\triangleq[c_1,c_2,\ldots,c_{\hat{m}}]^T,$$

and

$$\Psi(t)\triangleq\left[\psi_1(t),\psi_2(t),\ldots,\psi_{\hat{m}}(t)\right]^T. \tag{11}$$

Similarly, an arbitrary function of two variables $u(x,t)$ defined over $[0,1]\times[0,1]$, may be expanded by the LWs as follows:

$$u(x,t)\simeq\sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}}u_{ij}\psi_i(x)\psi_j(t)=\Psi^T(x)U\Psi(t), \tag{12}$$

where $U=[u_{ij}]$ and $u_{ij}=(\psi_i(x),(u(x,t),\psi_j(t)))$.

The convergence of the LWs expansion in two dimensions is investigated in the following theorems:

Theorem 3.1. *(See [32].) If the sum of the absolute value of the LWs coefficients of a continuous function $u(x,t)$ form a convergent series, then the LWs expansion is absolutely uniformly convergent, and convergent to the function $u(x,t)$.*

Theorem 3.2. *(See [32].) If a continues function $u(x,t)$ has bounded mixed fourth partial derivative $|\frac{\partial^4u(x,t)}{\partial x^2\partial t^2}|\leq\hat{M}$, then the LWs expansion of the function converges uniformly to the function and also*

$$|u_{ij}|\leq\frac{12\hat{M}}{(2n_1)^{\frac{5}{2}}(2n_2)^{\frac{5}{2}}(2m_1-3)^2(2m_2-3)^2}, \tag{13}$$

where $i=M(n_1-1)+m_1+1$ and $j=M(n_2-1)+m_2+1$.

By taking the collocation points $(t_i=\frac{i}{\hat{m}-1}, i=0,1,\ldots,\hat{m}-1)$ into (11), we define the LWs matrix $\Phi_{\hat{m}\times\hat{m}}$ as:

$$\Phi_{\hat{m}\times\hat{m}}\triangleq\left[\Psi(0),\Psi\left(\frac{1}{\hat{m}-1}\right),\ldots,\Psi(1)\right]. \tag{14}$$