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Optimal values of bipartite entanglement in a tripartite system

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ABSTRACT

For a general tripartite system in some pure state, an observer possessing any two parts will see them in a mixed state. By the consequence of Hughston–Jozsa–Wootters theorem, each basis set of local measurement on the third part will correspond to a particular decomposition of the bipartite mixed state into a weighted sum of pure states. It is possible to associate an average bipartite entanglement ($\bar{\mathcal{S}}$) with each of these decompositions. The maximum value of $\bar{\mathcal{S}}$ is called the entanglement of assistance (E_A) while the minimum value is called the entanglement of formation (E_F). An appropriate choice of the basis set of local measurement will correspond to an optimal value of $\bar{\mathcal{S}}$; we find here a generic optimality condition for the choice of the basis set. In the present context, we analyze the tripartite states W and GHZ and show how they are fundamentally different.

1. Introduction

Besides its fundamental importance in interpreting and understanding quantum mechanics, the quantum entanglement has attracted an immense interest in recent times because of its potential to play a significant role in modern technology. In addition, the quantum entanglement has now become a powerful tool to study the quantum many-body systems [1]. To be able to use the entanglement effectively and efficiently, it is necessary to characterize and quantify it by meaningful ways. There are many entanglement measures proposed for this purpose. In the axiomatic approach, there are some conditions to be satisfied for an entanglement measure to be a proper monotone [2,3]. For pure bipartite state, the *von Neumann entropy* is a suitable and widely used entanglement measure. Unfortunately it is not a good measure for non-pure or mixed bipartite state. For the mixed states, two popular entanglement measures are the *concurrence* [4] and *negativity* [5]. These two measures are reliable for distinguishing entangled states from separable states respectively for 2×2 system, and 2×2 and 2×3 systems. Generalization of these measures to higher dimensions is possible, but most of the time they are not satisfactory and unique [2,6,7].

The entanglement of formation (E_F^∞) [8] for a mixed state is a more general concept than the concurrence. For 2×2 system, the concurrence is monotonically related to E_F^∞ [9]. Even though E_F^∞ is a good measure for any bipartite mixed state, it is extremely difficult to calculate. The reason can be understood from the definition of the quantity, as shown below,

$$E_F^\infty(\rho) = \inf\Big\{\sum_i p_i \mathcal{S}_i\big(|\psi_i\rangle\big) : p_i \geq 0, \sum_i p_i = 1; \\ \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|\Big\}. \quad (1)$$

Here $\mathcal{S}_i$ can be any good entanglement measure for the pure bipartite state $|\psi_i\rangle$ (e.g. it can be von Neumann entropy). Basically, to find E_F^∞ one has to do minimization of an average quantity over infinite possible decomposition of the given mixed state (ρ).

The entanglement of assistance (E_A^∞) is another measure for an arbitrary bipartite mixed state [10]. Though it is not a proper monotone [11], it got importance due to its possible application in quantum technology. It is defined as the maximum possible average entropy between two parties, as shown below,

$$E_A^\infty(\rho) = \sup\Big\{\sum_i p_i \mathcal{S}_i\big(|\psi_i\rangle\big) : p_i \geq 0, \sum_i p_i = 1; \\ \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|\Big\}. \quad (2)$$

Quantifying and classifying multipartite entanglement is a very hard problem; it is now generally accepted that a single number is not enough for the purpose [2]. In this work we concentrate on studying a general tripartite system in a pure state. First we

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note that, according to the Hughston–Jozsa–Wootters (HJW) theorem [12], any finite decomposition compatible with a given mixed bipartite state can be created by a local measurement on a third part added in a pure state with the bipartite system. For a general tripartite system (A–B–C) in pure state, this theorem implies that, any local measurement on a part (C) will correspond to a particular decomposition of the bipartite mixed state that the other two parts (A–B) are effectively in. It may be here worth mentioning that all the quantum measurements in this work are considered to be non-selective projective-type (von Neumann measurement). One can associate an average entanglement ($\bar{\mathcal{S}}$) with every decomposition of a mixed state; very often this quantity is called the entanglement 'localized' between two parts of a tripartite system by doing a local measurement on the third part. This average quantity is always positive and have an upper bound for a finite dimensional system. This implies that $\bar{\mathcal{S}}$ will have optimal (maximum/minimum) values and as a consequence of the HJW theorem, these optimal values will correspond to some basis sets of measurement. In other words, since the decomposition of mixed state changes with the measurement basis, it is possible to get optimal values of the quantity $\bar{\mathcal{S}}$ by choosing appropriate measurement basis sets. The maximum and minimum values of $\bar{\mathcal{S}}$ are termed respectively as the entanglement of assistance (E_A) and the entanglement of formation (E_F) for the given tripartite pure state ($|\Psi\rangle$). These two quantities are given by following equations,

$$E_F(|\Psi\rangle) = \inf\left\{ \sum_i p_i \mathcal{S}_i(|\psi_i\rangle^{AB}) : p_i \geq 0, \sum_i p_i = 1; \right.$$
$$\left. |\Psi\rangle = \sum_i \sqrt{p_i}|\phi_i\rangle^C |\psi_i\rangle^{AB} \right\}. \tag{3}$$

$$E_A(|\Psi\rangle) = \sup\left\{ \sum_i p_i \mathcal{S}_i(|\psi_i\rangle^{AB}) : p_i \geq 0, \sum_i p_i = 1; \right.$$
$$\left. |\Psi\rangle = \sum_i \sqrt{p_i}|\phi_i\rangle^C |\psi_i\rangle^{AB} \right\}. \tag{4}$$

Here $|\phi_i\rangle^C$s form an orthonormal basis set of C and this set is used as a basis set for local measurement. After a measurement, parts A and B jointly assume pure state $|\psi_i\rangle^{AB}$ with probability p_i. In general $|\psi_i\rangle^{AB}$s are not orthogonal to each other (see Section 2).

Here it may be briefly mentioned that though there are some attempts to generalize the definition of the entanglement of formation to the multipartite systems (see for example [13]), but proper generalization is not possible until we have a clear notion of maximally entangled multipartite states [6,14]. This notion is still lacking; situation here is even more complicated due to the existence of different classes of multipartite states (for example, there are two non-interconvertible classes of tripartite states [15]). The definitions of the entanglement of formation and the entanglement of assistance given in this work for the pure tripartite states (cf. Eqs. (3) and (4)) are not an attempt to define any tripartite entanglement monotones by generalizing the corresponding bipartite monotones; these definitions would be though very useful and will serve us two purposes. Besides giving some important informations about the given tripartite state (see Section 3), they will help us calculate E_A^∞ and E_A^∞ for a given bipartite mixed state by using the concept of *ancilla* (as deliberated below).

It is known that, a person possessing two parts (A and B) of a tripartite system (which is in a pure state $|\Psi\rangle$) will only see a reduced state $\rho^{AB} = \mathrm{Tr}_C(|\Psi\rangle\langle\Psi|)$. As a consequence of the HJW theorem, each basis set of measurement of C corresponds to a particular decomposition of the mixed state ρ^{AB}. The number of pure states appearing in a decomposition cannot exceed the basis set dimension of C (say, D_C) [12]. On the other hand, the number of terms in an unrestricted decomposition of a given mixed bipartite state (without reference to C or any pure tripartite state) can be in principle any large number. According to the variational principle, the unrestricted minimization of the average quantity $\bar{\mathcal{S}}$ will give a lower value than the restricted minimization of the quantity. This implies that, the entanglement of formation (E_F) of a pure tripartite state as defined in Eq. (3) is higher than E_F^∞ for the corresponding reduced state (ρ^{AB}), i.e., $E_F^\infty(\rho^{AB}) \leq E_F(|\Psi\rangle)$. Following a similar line of argument we can also say that $E_A^\infty(\rho^{AB}) \geq E_A(|\Psi\rangle)$.

In general finding E_F^∞ or E_A^∞ for an arbitrary bipartite mixed state is difficult. As we mentioned before, in principle there can be any number of terms in an unrestricted decomposition of a mixed state. This makes even numerical calculations of the quantities really hard. It is though speculated that, to evaluate E_F^∞ or E_A^∞ for a mixed bipartite state ρ, it is enough to consider only a finite number of terms in the decomposition of ρ. For example, it was proved that it is sufficient to consider only r^2 terms in a decomposition to find E_F^∞, where r is the rank of the mixed state [16]. In fact it turned out that, for 2×2 systems it is enough to consider only four states for the purpose [17,18]. Therefore to find E_F^∞ or E_A^∞ for a given mixed state, we can start with a pure tripartite state for which the reduced state ρ^{AB} is the same as the given mixed state. The value of E_F (E_A) obtained from the tripartite state would not be the same as E_F^∞ (E_A^∞) if the number of terms required for the optimization is more than D_C (basis set dimension of C). To increase the basis set dimension, one can add a suitable *ancilla* to the third part C and do a joint measurement [12].

Since the optimization process to find E_F or E_A is very demanding, to help doing that, we derive in this paper an *optimality condition*. The basis set of measurement which satisfies this condition will correspond to an optimal value of the average entropy $\bar{\mathcal{S}}$.

In the last part of this paper, we analyze two tripartite states W and GHZ, and show how they are fundamentally different in the present context.

2. The optimality condition

Let A, B and C be three parts of a tripartite system in the pure state $|\Psi\rangle$. A local quantum measurement on C by some basis set would result in S ($= A + B$) assuming different pure states with appropriate probabilities.

When expressed in the product basis states of C and S, the given tripartite state becomes,

$$|\Psi\rangle = \sum_{i,j=1,1}^{D_C,D_S} g_{i,j}|\xi_i\rangle^C|\phi_j\rangle^S. \tag{5}$$

Here $|\xi_i\rangle^C$s ($|\phi_i\rangle^S$s) are some orthonormal basis vectors of the state space of C (S) with dimensionality D_C (D_S). This state is assumed to be normalized: $\sum_{i,j=1,1}^{D_C,D_S} g_{i,j}g_{i,j}^* = 1$. Let us now rewrite this state in the following special form,

$$|\Psi\rangle = \sum_{i=1}^{D} \sqrt{p_i}|\xi_i\rangle^C|\xi_i\rangle^S, \tag{6}$$

with $p_i = \sum_{j'=1}^{D_S} g_{i,j'}g_{i,j'}^*$ and $|\xi_i\rangle^S = \sum_{j=1}^{D_S} \frac{g_{i,j}}{\sqrt{p_i}}|\phi_j\rangle^S$. Here the summation runs over nonzero p_i's, numbering D ($\leq D_C$). In general, states $|\xi_i\rangle^S$s are not orthogonal (but they all are normalized). The operational interpretation of the later expression of the state $|\Psi\rangle$ given in Eq. (6) is that, if we perform a local quantum measurement on C by the basis set $\{\xi^C\}$, the state $|\Psi\rangle$ will collapse and we will get S in different pure states $|\xi_i\rangle^S$s with corresponding probabilities p_i's.