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A note on finite-scale Navier–Stokes theory: The case of constant viscosity, strictly adiabatic flow

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ABSTRACT

We investigate the "piston problem" for the case of a viscous, but non-thermally conducting, gas with constant transport coefficients under the recently introduced generalization of the Navier–Stokes (NS) equations known as the finite-scale Navier–Stokes (FSNS) equations. Along with determining and analyzing the integral curves of the resulting kink-type traveling wave solutions (TWS)s, the present study also reveals the importance of the bulk viscosity vis-a-vis this special case of FSNS theory and highlights the impact that averaging has on the structure of the shock profile.

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1. Introduction

The FSNS equations were put forth by Margolin [9] in 2009. They are obtained from the classical NS equations via the (spatial) averaging transform

$$\widehat{\psi}(x,t) = L^{-1} \int_{x-L/2}^{x+L/2} \psi(\mathrm{x},t)\mathrm{dx}, \tag{1}$$

where ψ represents each of the terms in the NS system and the constant $L(>0)$ is the averaging length scale, and use of the closure theorem

$$\widehat{\mathcal{A}\mathcal{B}} = \widehat{\mathcal{A}}\widehat{\mathcal{B}} + \left(\frac{L^2}{12}\right)\widehat{\mathcal{A}}_x\widehat{\mathcal{B}}_x + O\big(L^4\big), \tag{2}$$

where $\widehat{\mathcal{A}}_x := \partial\widehat{\mathcal{A}}/\partial x$ and terms $O(L^4)$ are neglected in the finalized equations, to handle products; see also Refs. [10,11] and those therein. What distinguishes Margolin's approach from other averaging schemes which have been applied to the NS equations is the fact that L depends on the *observer*. That is, L is not inherent to the physical process under consideration; instead, it arises from the means by which the flow is probed or modeled [10, §2]. This interpretation of L also distinguishes the finite-scale formulation from other theories of generalized continua.[1]

To date, the only analytical solution of the FSNS equations that has been obtained is one of the traveling wave type for the case of a gas in which $\mu \propto \rho$, $\mu_{\mathrm{B}} = 0$, and $K = 0$; see Ref. [24, Eq. (39)], which satisfies Ref. [11, Eq. (24)] under appropriate rescaling. Here, $\rho(>0)$ is the mass density[2] of the gas; μ and μ_{B} are, respectively, its coefficients of shear and bulk viscosity; and K is the coefficient

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[1] That is, generalizations of classical continuum mechanics that seek to capture the effects of media micro-structure via the introduction of one of more length-scale parameters; see, e.g., Refs. [5,12,16–18] and those therein.

[2] Hereafter, in keeping with the notation scheme of Ref. [11], ρ will be used to represent $\widehat{\rho}$.

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of thermal conductivity. Clearly, then, there is a need to investigate the nature of the solutions, in particular, those of the traveling wave type, admitted by the FSNS system when different constitutive laws for the transport coefficients are assumed.

The aim of this Note is to probe the FSNS system along the lines just described; specifically, in the context of the piston problem,[3] we determine and analyze the TWS for a gas in which $\mu(>0)$, $\mu_B(\geq 0)$ are constants, but $K=0$. This special case of transport coefficients is one of the few for which the (1D) NS system admits an exact solution to the piston problem; see Hayes [6] for a more complete listing. As such, the present study generalizes those carried out by Lord Rayleigh [15] and Taylor [20], both in 1910, to include the influence of the averaging length scale, L. The work presented below demonstrates the importance of the bulk viscosity vis-a-vis this special FSNS theory; it also highlights the impact that averaging has on the structure of the shock profile.

Remark 1. The special case of zero thermal conductivity, but nonzero viscosity, has been termed *strictly adiabatic* by von Mises [22].

2. Traveling wave analysis: finite-scale theory

Since the (1D) FSNS system has been derived and discussed in several other works (see, e.g., Refs. [9,11]) it will not be restated here. Instead, we begin with Margolin and Vaughan's [11] analysis of the piston problem and take the averaged field variables to be functions of the similarity (or wave) variable $y=x-vt$, where the speed v of the waveform we seek is a positive constant. The finite-scale versions of the 1D continuity, momentum, and energy[4] equations are thus reduced to [11, §3]

$$\frac{\mathrm{d}}{\mathrm{d}y}(-v\rho+u\rho)=0, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(-vu\rho+u^2\rho+P_\mathrm{c}\right)=0, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(-vU\rho-\frac{1}{2}\rho u^2 v+\rho uU+\frac{1}{2}\rho u^3+P_\mathrm{c}u\right)=0, \tag{5}$$

respectively, where we recall that the thermal conductivity of the gas is assumed to be negligibly small. Here, to accommodate our assumption of constant viscosity, we have modified Eqs. (13) and (19) of Ref. [11] to read

$$P_\mathrm{c}=(\gamma-1)\rho U+\rho\chi_\mathrm{c}\big(u'\big), \tag{6}$$

$$\chi_\mathrm{c}\big(u'\big):=A\big(u'\big)^2-\mu_\mathrm{L}\rho^{-1}u', \tag{7}$$

respectively, and we refer the reader to Ref. [11, Eq. (7)] for the defining relations of u and U. In Eqs. (6) and (7), $\gamma\in(1,5/3]$ denotes the ratio of specific heats [21], which we take to be a constant; $A:=L^2/12$ is, of course, a constant as well; $\mu_\mathrm{L}(>0)$, termed the longitudinal coefficient of viscosity [6], is defined as $\mu_\mathrm{L}:=\mu(\frac{4}{3}+\mu_\mathrm{B}/\mu)$, where we note that $\mu_\mathrm{B}=0$ for monatomic gases [13,21]; and a prime denotes $\mathrm{d}/\mathrm{d}y$.

2.1. Associated ODE

Now integrating Eq. (5) and then using Eq. (6) to eliminate P_c from the result yields, after solving for the constant of integration $\mathcal{K}_1$,

$$\rho U\big[(v-u)-(\gamma-1)u\big]+\frac{1}{2}\rho(v-u)u^2-\rho u\chi_\mathrm{c}\big(u'\big)=v\rho_0U_0, \tag{8}$$

where $\mathcal{K}_1=v\rho_0U_0$ and a zero subscript denotes the (constant) equilibrium state value of the quantity to which it is attached. In turn, eliminating the product ρU between Eqs. (8) and (A.5), the latter of which we first re-express in terms of $\chi_\mathrm{c}(u')$ as

$$\rho U=\rho_0U_0+(\gamma-1)^{-1}\big[\rho_0vu-\rho\chi_\mathrm{c}\big(u'\big)\big], \tag{9}$$

and then solving for $\chi_\mathrm{c}(u')$, we obtain the associated ODE of the present study, specifically,

$$A\big(u'\big)^2-\eta v^{-1}(v-u)u'=\frac{u}{v}\big[\big(v^2-c_0^2\big)-\beta vu\big]. \tag{10}$$

Here, ρ has been eliminated from the LHS using Eq. (A.4); $\eta=\mu_\mathrm{L}/\rho_0$ is the kinematic longitudinal viscosity coefficient, where the constant ρ_0 denotes the value of the mass density ahead of the shock; we have used the fact that $\beta=(\gamma+1)/2$ in the case of perfect gases, where $\beta(>1)$ is the coefficient of nonlinearity; and we note that the square of the adiabatic sound speed ahead of the shock is given by $c_0^2=\gamma(\gamma-1)U_0$, where the constant U_0 denotes the value of the specific internal energy ahead of the shock [11, p. 66].

Seeking integral curves in the form of kinks [1], we impose the asymptotic conditions

$$\lim_{y\to\mp\infty}u(y)=\{u_\mathrm{p},0\}, \tag{11}$$

[3] The mathematical treatment of this problem for a dissipative gas can be traced back to an 1870 paper by Rankine [14] in which he examined the case of a gas whose only loss mechanism is its ability to conduct heat; see also Lamb [7, Art. 284] and Lord Rayleigh [15].

[4] Ref. [11, Eq. (12)] contains a misprint; see Appendix A.