Physics Letters A 374 (2010) 1113–1116

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Dispersion relation for electromagnetic waves in anisotropic media

Yakov Itin [a,b,*]

[a] Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel
[b] Jerusalem College of Technology, Jerusalem, Israel

ARTICLE INFO

Article history:
Received 21 September 2009
Received in revised form 23 December 2009
Accepted 28 December 2009
Available online 7 January 2010
Communicated by P.R. Holland

Keywords:
Electrodynamics
Relativity
Constitutive law
Anisotropic media

ABSTRACT

Electromagnetic wave propagation in anisotropic dielectric media with two generic matrices ε^{ij} and μ^{ij} of permittivity and permeability is studied. In the framework of a metric-free electrodynamics approach, a compact tensorial dispersion relation is derived. The derivation does not require the corresponding matrices to be symmetric, positive definite, nor even invertible. The resulting formula is useful for a theoretical and experimental study of electromagnetic wave propagation in a wide class of linear media.

1. Introduction

The development of the modern microscopic technology (nanotechnology) provides a possibility to manufacture materials of rather non-ordinary electromagnetic parameters. This situation calls for a theoretical investigation of electromagnetic wave propagation in media with a generic constitutive law.

Undoubtedly, wave propagation is a most generic physical phenomena which joins the classical and quantum field theories. Moreover it emerges in a broad class of theoretical and experimental subjects, particularly in the high energy physics, general relativity, astrophysics, materials science and the plasma physics. The theoretical issues connected to the wave propagation phenomena yield a class of intriguing mathematical physics problems.

A wide class of media is characterized by a linear constitutive law and in general can be described by four 3×3 matrices. Two of these matrices, ε^{ij} and μ^{ij} (permittivity and permeability matrices), describe the pure electric and the magnetic properties of matter, respectively. Two additional matrices describe relatively smaller electric–magnetic cross-term effects.

The ordinary textbook's description of a medium with two anisotropic matrices ε^{ij} and μ^{ij} is based on their diagonalization [1–3]. This algebraic procedure is always possible in the case when both matrices are symmetric and one of them is positive definite. Even with this restriction, the corresponding dispersion relation was obtained in a rather complicated form. Moreover, it is clear that the diagonalization technique is not applicable in a general case when both matrices ε^{ij} and μ^{ij} are not symmetric nor positive definite. Such extensions of the standard electromagnetic materials properties are not only of a theoretical interest. In fact, the medium with negative permittivity and permeability parameters serves as a theoretical basis for recently manufactured metamaterials. The non-symmetric matrices are necessary for description of the medium which parameters are modified by external electromagnetic fields (the magnetized ferrite). The medium with non-invertible matrices is recently discussed in form of a perfect electromagnetic conductor.

In the current Letter, we study the wave propagation in a generic medium in a framework of premetric electrodynamics approach [4–9]. Our final result is a compact form of the dispersion relation for electromagnetic waves in an anisotropic medium. For two generic matrices ε^{ij} and μ^{ij}, it is given by the expression

$$\boxed{w^4 - 2\left(\psi^{ij}k_ik_j\right)w^2 + \frac{\varepsilon^{ij}k_ik_j}{\det\varepsilon}\,\frac{\mu^{mn}k_mk_n}{\det\mu} = 0.} \tag{1.1}$$

Here the matrix ψ^{ij} is defined as

$$\psi^{ij} = \frac{1}{2}\epsilon^{imn}\epsilon^{jpq}\varepsilon^{-1}_{nq}\mu^{-1}_{mp}. \tag{1.2}$$

The organization of the Letter is as follows: In the next section, the metric-free electrodynamics notation is recalled. In Section 3, the covariant metric-free form of the dispersion relation is represented. The main results are given in Section 4 where several compact forms of the generic dispersion relation and some straightforward consequences of them are derived. In Section 5,

* Address for correspondence: Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel.
E-mail address: itin@math.huji.ac.il.

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2009.12.071

the examples for isotropic, diagonal anisotropic, and non-diagonal (magnetized ferrite) media are represented.

2. Anisotropic media in the metric-free description

Let us start with a metric-free four-dimensional system of Maxwell equations

$$\varepsilon^{\alpha\beta\gamma\delta}F_{\beta\gamma,\delta}=0, \qquad \mathcal{H}^{\alpha\beta}{}_{,\beta}=4\pi\mathcal{J}^{\alpha}. \tag{2.1}$$

It includes an antisymmetric tensor of the *field strength* $\mathcal{F}_{\alpha\beta}$ and an antisymmetric tensor density of the *field excitation* $\mathcal{H}^{\alpha\beta}$. The coordinate indices are denoted by Greek letters which run over the range of $\alpha,\beta,\ldots=0,1,2,3$, the comma denotes the partial derivatives relative to the coordinates $\{x^0,x^1,x^2,x^3\}=\{t,x,y,z\}$. In the sequel, the Roman indices will be used for the spatial coordinates, $i,j,\ldots=1,2,3$. The four-dimensional Levi-Civita's permutation pseudo-tensor $\epsilon^{\alpha\beta\gamma\delta}$ is normalized as $\epsilon^{0123}=1$, while $\epsilon_{0123}=-1$.

The $(1+3)$-decomposition of the field tensors reads

$$E_i=F_{0i}, \qquad B^i=-\frac{1}{2}\varepsilon^{ijk}F_{jk}, \tag{2.2}$$

$$D^i=\mathcal{H}^{0i}, \qquad H_i=\frac{1}{2}\varepsilon_{ijk}\mathcal{H}^{jk}. \tag{2.3}$$

The electric current is given by $\mathcal{J}^0=\rho$, and $\mathcal{J}^i=j^i$. In this notation, the system (2.1) is rewritten in the ordinary three-dimensional form of Maxwell equations

$$\operatorname{div}\mathbf{B}=0, \qquad \operatorname{curl}\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0, \tag{2.4}$$

$$\operatorname{div}\mathbf{D}=4\pi\rho, \qquad \operatorname{curl}\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=4\pi\mathbf{j}. \tag{2.5}$$

For a dielectric medium, two antisymmetric tensor fields are assumed to be linearly related one to another

$$\mathcal{H}^{\alpha\beta}=\frac{1}{2}\chi^{\alpha\beta\gamma\delta}\mathcal{F}_{\gamma\delta}. \tag{2.6}$$

The *constitutive tensor* $\chi^{\alpha\beta\gamma\delta}$ is antisymmetric in two pairs of indices, so it has, in general, 36 independent components. Such a generic constitutive tensor can be represented by four three-dimensional matrices of 9 independent components. This representation can be written in the form

$$\chi^{\alpha\beta\gamma\delta}=\begin{pmatrix}\varepsilon^{ij} & \gamma^i{}_j\\ \tilde{\gamma}^i{}_j & \pi_{ij}\end{pmatrix}. \tag{2.7}$$

We restrict ourselves to an electromagnetic medium which is described by two tensors ε^{ij} and π_{ij}. Two additional tensors $\gamma^i{}_j$ and $\tilde{\gamma}^i{}_j$ represent the electric–magnetic cross-terms, which are relatively small for most types of the dielectric materials. In this Letter, they are taken to be equal to zero. We will consider, however, some type of a *generalized anisotropic medium*. In particular, we will not require the matrices ε^{ij} and π_{ij} to be symmetric, positive definite, nor even invertible. Consequently we will use a constitutive tensor of 18 independent components

$$\chi^{\alpha\beta\gamma\delta}=\begin{pmatrix}\varepsilon^{ij} & 0\\ 0 & \pi_{ij}\end{pmatrix}. \tag{2.8}$$

Note the relations:

$$\chi^{0i0j}=\varepsilon^{ij}, \qquad \chi^{ijkl}=-\epsilon^{ijm}\epsilon^{kln}\pi_{mn}, \tag{2.9}$$

where the three-dimensional Levi-Civita's permutation pseudo-tensor ϵ^{ijk} is normalized as $\epsilon^{123}=1$. In three-dimensional form, the corresponding constitutive relation is given by

$$D^i=\varepsilon^{ij}E_j, \qquad H_i=\pi_{ij}B^i. \tag{2.10}$$

For a regular *impermeability* matrix π_{ij}, an inverse *permeability* matrix is defined —

$$\left(\pi^{-1}\right)^{ij}=\mu^{ij}. \tag{2.11}$$

With this notation, the constitutive relation takes the ordinary form

$$D^i=\varepsilon^{ij}E_j, \qquad B^i=\mu^{ij}H_j. \tag{2.12}$$

3. A general dispersion relation

Recently, a covariant dispersion relation for a generic constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ was studied intensively, see [5,6]. Here we briefly recall the necessary notation and the main stages of the derivation as it is given in [9].

Our aim is to establish the necessary conditions for existence of physically non-trivial solutions of the source-free system

$$\epsilon^{\alpha\beta\gamma\delta}F_{\beta\gamma,\delta}=0, \qquad \chi^{\alpha\beta\gamma\delta}F_{\beta\gamma,\delta}=0. \tag{3.1}$$

Let the ordinary conditions of the geometric optics approximation be accepted. In particular, we consider the media parameters encoded in $\chi^{\alpha\beta\gamma\delta}$ as varying slowly relative to the change of the electromagnetic field.

The first equation of (3.1) has a standard solution in term of the vector potential A_α

$$F_{\alpha\beta}=\frac{1}{2}(A_{\alpha,\beta}-A_{\beta,\alpha}). \tag{3.2}$$

Consequently, the second equation of (3.1) takes the form

$$\chi^{\alpha\beta\gamma\delta}A_{\gamma,\beta\delta}=0. \tag{3.3}$$

Let us look for a solution of this equation in the form of a monochromatic wave ansatz

$$A_\alpha=a_\alpha e^{iq_\beta x^\beta}. \tag{3.4}$$

We substitute this ansatz into (3.3) and treat the amplitude of the field a_α and the wave covector q_β as slowly varying functions of a spacetime point. Consequently, we come to an algebraic system

$$M^{\alpha\delta}a_\delta=0 \tag{3.5}$$

with a characteristic matrix

$$M^{\alpha\delta}=\chi^{\alpha\beta\gamma\delta}q_\beta q_\gamma. \tag{3.6}$$

This matrix evidently satisfies the relations

$$M^{\alpha\delta}q_\alpha=0, \qquad M^{\alpha\delta}q_\delta=0. \tag{3.7}$$

These relations have a clear physical meaning. The first equation represents the *charge conservation law*, while the second one means that the ansatz (3.4) with $q_\alpha\sim a_\alpha$ is a solution of (3.5). Certainly this solution does not have a physical meaning, because it corresponds to a zero value of the field $F_{\alpha\beta}$, i.e., it is related to the *gauge invariance* of the field equations.

Thus we are looking for solutions of the system (3.5) constrained by the relations (3.6). In the matrix language, these relations mean that the columns and the rows of the matrix $M^{\alpha\delta}$ are linearly dependent, i.e., the matrix is singular. Consequently, our system always has a non-zero solution. However, due to the gauge invariance, we need more of that. In fact, we are looking for an additional linear independent solution. Only this one will be of a physical meaning.

It is an algebraic fact, that a linear system has two independent solution only if the adjoint of the characteristic matrix is equal to zero. So we come to an equation

$$(adj\,M)_{\alpha\beta}=0. \tag{3.8}$$