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Exactly solvable diffusion models in the framework of the extended supersymmetric quantum mechanics

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ABSTRACT

This work is aimed at demonstrating the possibility to construct new exactly-solvable stochastic systems by use of the extended supersymmetric quantum mechanics ($N = 4$ *SUSY QM*) formalism. A feature of the proposed approach consists in the fact that obtained new potentials, which enter the Langevin equation, and so probability densities have a parametric freedom. The latter allows one to change the form of potentials without changing the temporal behavior of the probability densities.

1. Introduction

Stochastic processes managed by various types of noise may be found in many physical systems, and in a wide range from micro scales up to the Universe. Such systems have received a lot of attention, mostly due to theoretical predictions and experimental verifications of the “stochastic resonance” phenomenon, the directional currents of particles in the so-called ratchet systems and so on. Today one may encounter a lot of situations in which the stochastic resonance is realized in different ways. Several methods of studies of this phenomenon have been developed up to the date. We should note, however, that the complexity of the considered systems does not allow to apply only analytical methods, so in the most cases complex numerical simulations are required. One of the basic equations which are actively exploited as in analytical as well as in the numerical studies of the stochastic dynamics is the Fokker–Planck (FP) equation. Applications and solutions of the FP equation have been extensively discussed in literature (see, e.g. [1,2]). One of the useful methods of solving the FP equation is the eigenfunction expansion method. This method is quite similar to the bound state expansion applying to the Schrödinger equation, and it plays the same role for the FP equation, especially when a formal similarity between two equations is taken into account. This similarity will be actively exploited in what follows, and now let us make the following remark. Sometimes, for very special reasons, it becomes possible to completely resolve a quantum mechanical problem. Formally, the eigenfunctions and the eigenvalues of the quantum mechanical Hamiltonian could be related to that of the corresponding FP equation. Hence, we have a correspondence between quantum mechanics and classical stochastic dynamics, so finding a new exactly solvable quantum mechanical problem, the number of which was substantively increased last time [3,4], will generate a new well-controlled dynamical stochastic system. As an example of intertwining between quantum mechanics and the stochastic dynamics it is worth mentioning the construction a nonlinear model of diffusion in the bistable stochastic system realized in [5,6]. The formalism of the Darboux transformation used in [5,6] later became the one of the methods for construction of isospectral Hamiltonians in supersymmetric quantum mechanics (*SUSY QM*).

The aim of this Letter is to demonstrate the possibility to construct new exactly solvable models of stochastic systems in the framework of the extended supersymmetric quantum mechanics ($N = 4$ *SUSY QM*). The distinctive feature of the proposed approach is the existence of a parametric freedom in probability densities and new potentials entering the Langevin equation, that makes possible to change the shape of potentials and densities without changing the time dependence of the probability density. We believe that extending the class of exactly solvable diffusion models serves to the construction of more relevant approximations for the real processes.

The organization of the Letter is as follows. In the next section we give basic facts on the structure of $N = 4$ *SUSY QM*. Here we closely follow [7,8] and consider the procedure of constructing the isospectral Hamiltonians. A parametric freedom which is a feature of such a model will be under the focus. In Section 3 we construct new models of stochastic systems related to the supermultiplet of the $N = 4$ *SUSY QM* isospectral Hamiltonians. We give

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the explicit expressions for the probability densities and the potentials entering the Langevin equation. The proposed scheme of getting new models of stochastic systems is considered in details in Section 4 where we discuss its application to the well-known Ornstein–Uhlenbeck process. The early obtained stochastic systems for this process were related to a particle motion in the strong friction regime with the Gaussian noise and the potential of the harmonic oscillator and its inverse. We obtain the stochastic systems with an essentially non-linear potential the probability densities of which possess the same temporal behavior as in the above mentioned stochastic systems. Our conclusions are collected in the last section.

2. $N = 4$ *SUSY QM*

The FP equation is equivalent to the Langevin equation, however the Fokker–Planck is used more widely in physics, since it is formulated in more common for the probability density $m_t^{\pm}(x, x_0)$ language [4]:

$$\frac{\partial}{\partial t} m_t^{\pm}(x, x_0) = \frac{D}{2}\frac{\partial^2}{\partial x^2} m_t^{\pm}(x, x_0) \mp \frac{\partial}{\partial x}\Phi(x) m_t^{\pm}(x, x_0),$$

$$m_{t=0}^{\pm}(x, x_0) = \langle \delta(x - x_0)\rangle, \qquad U_{\pm}(x) = \pm\int_0^x dz\, \Phi(x)$$

where $U_{\pm}(x)$ is a potential entering Langevin equation. The Fokker–Planck equation describes the stochastic dynamics of particles in potentials $U_+(x)$ and $U_-(x) = -U_+(x)$. Substituting

$$m_t^{\pm}(x, x_0) = \exp\left\{-\frac{1}{D}\left[U_{\pm}(x) - U_{\pm}(x_0)\right]\right\} K_{\pm}(x, t)$$

the FP equation transforms into Schrödinger equation with imaginary time:

$$-D\frac{\partial}{\partial t} K_{\pm}(x, t) = H_{\pm} K_{\pm},$$

$$K_{\pm}(x, t) = \langle x|\exp\left\{-\frac{tH_{\pm}}{D}\right\}|x_0\rangle,$$

$$H_{\pm} = -\frac{D^2}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left[\Phi^2(x) \pm D\Phi'(x)\right] \tag{1}$$

in which diffusion constant D could be treated as a “Planck constant”, while $H_{\pm}$ has a form of $N = 2$ *SUSY QM* Hamiltonian. Eqs. (1) form the base of the eigenfunctions expansion method. Details of this method could be found in [1,4].

The utilization of the extended $N = 4$ *SUSY QM* formalism provides additional opportunities for construction of new exactly-solvable models of stochastic processes. Hamiltonian $H_{\sigma_1}^{\sigma_2}$ of $N = 4$ *SUSY QM* includes four isospectral Hamiltonians with wave functions connected by supersymmetry transformation. This allows the obtaining of new $K(x, t)$ and $U(x)$ using exactly solvable quantum-mechanical models.

Hamiltonian of $N = 4$ supersymmetric quantum mechanics has a form:

$$H_{\sigma_1,\sigma_2} = \frac{1}{2}\left(p^2 + V_2^2(x) + \sigma_3^{(1)} V_2^1(x)\right)$$
$$\equiv \frac{1}{2}\left(p^2 + V_1^2(x) + \sigma_3^{(2)} V_1^1(x)\right) \tag{2}$$

where we denote ($\hbar = m = 1$):

$$V_i(x) = W'(x) + \frac{1}{2}\sigma_3^{(i)}\frac{W''(x)}{W'(x)}$$

and $W(x)$ is a superpotential and $\sigma_3^{(i)}$—matrixes which commutate with each other and their eigenvalues are ± 1:

$$\sigma_3^{(1)} = \sigma_3 \otimes 1, \qquad \sigma_3^{(2)} = 1 \otimes \sigma_3.$$

Supercharges Q_i of extended supersymmetric quantum mechanics form an algebra:

$$\{Q_i, \bar{Q}_k\} = 2\delta_{ik} H, \qquad \{Q_i, Q_k\} = \{\bar{Q}_i, \bar{Q}_k\} = 0, \quad i, k = 1, 2$$

and have a form:

$$Q_i = \sigma_-^{(i)}\left(p + iV^{(i+1)}(x)\right), \qquad \bar{Q}_i = \sigma_+^{(i)}\left(p - iV^{(i+1)}(x)\right)$$

where $V^{(3)}(x) \equiv V^{(1)}(x)$, $\sigma_{\pm}^{(1)} = \sigma_{\pm} \otimes 1$, $\sigma_{\pm}^{(2)} = 1 \otimes \sigma_{\pm}$.

Hamiltonian and supercharges act on 4-dimensional internal space. At that Hamiltonian is diagonal on vectors $\psi_{\sigma_1}^{\sigma_2}(x, E)$, where σ_1, σ_2-eigenvalues of operators $\sigma_3^{(1)}$, $\sigma_3^{(2)}$. Supercharges $Q_i(\bar{Q}_i)$ act as a step-down (step-up) operators for indexes σ_1, σ_2.

Construction of isospectral Hamiltonians in the context of $N = 4$ *SUSY QM* is based on the fact that four Hamiltonians are joined to the super-multiplet. This procedure considered in details in [7,8]. Super-potential has the form:

$$W(x) = \frac{\sigma_1}{2}\ln\left(1 + \lambda\int_{x_i}^x dx'\left[\varphi\left(x', \varepsilon, c\right)\right]^{2\sigma_1\sigma_2}\right) \tag{3}$$

where $\varphi(x', \varepsilon, c)$ is the general solution of the subsidiary equation $H\varphi(x) = \varepsilon\varphi(x)$, ε—the so-called factorization energy and λ—arbitrary parameters conditioned by $-1 < \lambda$. In the case when $\varepsilon = E_0$ coincides with ground state energy of the initial Hamiltonian H_-^- with wave function $\psi_-^-(x, E_0)$, the connection between Hamiltonians of the super-multiplet and corresponding wave functions is following:

$$H_+^- = H_-^- - \frac{d^2}{dx^2}\ln\psi_-^-(x, E_0),$$

$$\psi_+^-(x, E) = \frac{1}{\sqrt{2(E - \varepsilon)}}\frac{Wr\{\psi_-^-(x, E_0)\psi_-^-(x, E)\}}{\psi_-^-(x, E_0)},$$

$$H_+^+ = H_-^- - \frac{d^2}{dx^2}\ln\left(1 + \lambda\int_{x_i}^x dx'\left[\psi_-^-\left(x', E_0\right)\right]^2\right),$$

$$\psi_+^+(x, E) = \psi_-^-(x, E) - \frac{\lambda\psi_-^-(x, E_0)}{1 + \lambda\int_{x_i}^x dx'\left[\psi_-^-(x', E_0)\right]^2}$$
$$\times\int_{x_i}^x dx'^2\, \psi_-^-\left(x', E_0\right)\psi_-^-\left(x', E\right) \tag{4}$$

where $Wr\{\}$ is a Wronskian. Wave functions $\psi_+^-(x, E)$ and $\psi_+^+(x, E)$ correspond to the states of Hamiltonian with $E_n > 0$. In addition, spectrum of H_+^+ has a state with $E_0 = 0$ (energy is counted of from ε) with wave function:

$$\psi_+^+(x, E_0) = (\lambda + 1)^{1/2}\frac{\psi_-^-(x, E_0)}{1 + \lambda\int_{x_i}^x dx'\left[\psi_-^-(x, E_0)\right]^2}. \tag{5}$$

As it could be seen from (4), parametric dependence on λ is only present in H_+^+ and $\psi_+^+(x, E)$. It should be notes that value of λ could not be fixed by normalization condition for wave functions. λ could take on the value $\lambda > -1$, due to singularities that emerge in opposite case and lead to violation of Hermitian conjugation of supersymmetry generators Q, $\bar{Q}$ and, hence, to singular potentials in Hamiltonians. Existence of parametric freedom is common when building isospectral Hamiltonians basing on different variants of the inverse scattering problem [9–11]. This is caused by