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Synchronized stability in a reaction–diffusion neural network model

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ABSTRACT

The reaction–diffusion neural network consisting of a pair of identical tri-neuron loops is considered. We present detailed discussions about the synchronized stability and Hopf bifurcation, deducing the non-trivial role that delay plays in different locations. The corresponding numerical simulations are used to illustrate the effectiveness of the obtained results. In addition, the numerical results about the effects of diffusion reveal that diffusion may speed up the tendency to synchronization and induce the synchronized equilibrium point to be stable. Furthermore, if the parameters are located in appropriate regions, multiple unstability and bistability or unstability and bistability may coexist.

1. Introduction

The study of the dynamics of the neural networks is an interdisciplinary matter, which has concentrated the interest of many researchers for the last decades [1] (e.g., mathematicians, physicists, computer scientists and so on). Since Marcus and Westervelet [2] incorporated a single time lag into the connection term of Hopfield's model, delays have been inserted into various simple neural networks, many authors have also investigated the dynamics of the neural networks of two or more neurons with delays, and have shown various types of dynamical behaviors (see, for example [3–8] and references therein). However, most of these work only considered the individual neural network but did not investigate the interactions between different neural networks.

As a matter of fact, neural networks consist of many nonlinear components which are interdependent and form a complex system with new emergent properties that are not held by each individual item in the system alone. Coupled networks, which are combined by subnetworks and each subnetwork has its own dynamical property, are ubiquitous and also common in many branches of science [9]. For instance, in order to describe the complicated interaction between billions of neurons in large neural networks, the neurons are often lumped into highly connected subnetworks and the brain organization can be viewed in gross sense as a number of local subnetworks coupled by long distance connections [10].

Recently, Shayer and Campbell [11] considered the following two coupled units

$$\begin{cases} \dfrac{dx_1}{dt} = -kx_1(t) + \beta \tanh\big(x_1(t-\tau_s)\big) \\ \qquad + a_{12}\tanh\big(x_2(t-\tau_2)\big), \\ \dfrac{dx_2}{dt} = -kx_2(t) + \beta \tanh\big(x_2(t-\tau_s)\big) \\ \qquad + a_{21}\tanh\big(x_1(t-\tau_1)\big). \end{cases}$$

They were interested in studying how time delays can affect not only the stability of fixed points of the network but also the bifurcation of new solutions when stability is lost. The authors [12] provided the stability and bifurcation of periodic solutions for a neural network with n elements where delays between adjacent units and external inputs are included, the particular cases $n = 2$ and $n = 3$ were discussed in detail.

The subnetwork of the coupling models both in [11] and [12] are single neuron. Song et al. [13] considered a neural network coupled by two sub-networks, each consisting of two neurons as follows

$$\begin{cases} \dfrac{du_1}{dt} = -u_1(t) + a_{12} f\big(u_2(t-\tau)\big) + \alpha f\big(u_4(t-\tau)\big), \\ \dfrac{du_2}{dt} = -u_2(t) + a_{21} f\big(u_1(t-\tau)\big), \\ \dfrac{du_3}{dt} = -u_3(t) + a_{12} f\big(u_4(t-\tau)\big) + \alpha f\big(u_2(t-\tau)\big), \\ \dfrac{du_4}{dt} = -u_4(t) + a_{21} f\big(u_3(t-\tau)\big), \end{cases}$$

the conditions ensuring the stability and direction of the Hopf bifurcation being determined. In [14], Campbell et al. studied the delayed neural network model coupled by a pair of Hopfield-like tri-neuron loops

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$$\begin{cases} \dfrac{dx_1}{dt} = -x_1(t) + \tanh\bigl(bx_3(t)\bigr), \\ \dfrac{dx_2}{dt} = -x_2(t) + \tanh\bigl(bx_1(t)\bigr), \\ \dfrac{dx_3}{dt} = -x_3(t) + \tanh\bigl(bx_2(t)\bigr) + c_1 \tanh\bigl(bx_6(t-\tau)\bigr), \\ \dfrac{dx_4}{dt} = -x_4(t) + \tanh\bigl(bx_6(t)\bigr), \\ \dfrac{dx_5}{dt} = -x_5(t) + \tanh\bigl(bx_4(t)\bigr), \\ \dfrac{dx_6}{dt} = -x_6(t) + \tanh\bigl(bx_5(t)\bigr) + c_2 \tanh\bigl(bx_3(t-\tau)\bigr). \end{cases}$$

They analyzed the roots of characteristic equation explicitly and specially investigated the local stability and bifurcation and observed some in-phase or anti-phase oscillations in numerical simulations. Recently, Hsu et al. [15] extended the results of Campbell et al. [14] to a delayed model comprised of a pair of loops each with n neurons. Based on [14] and [15], Peng and Song [16] studied a delayed neural network consisting of a pair of identical tri-neuron network loops with bidirectional coupling of all neurons between loops, while Yuan and Li [17] gave the explicit conditions ensuring the stability and direction of the Hopf bifurcation of the model in [16].

The rich dynamics arising from the interaction of simple units have been a source of interest for scientists modeling the collective behavior of real-life systems. Inspired by the above, a coupled network of dynamical systems can exhibit a range of interesting behavior, qualitatively very different from their behavior in isolation, such as synchronization [18,19], phase trapping, phase locking, and amplitude death.

Among them, synchronization, which is the phenomenon where systems, due to some kind of interaction, adjust their individual behavior in such a way that their behaviors become identical, has been causing researchers' wide focus [8,30,38] since the works by Pecora and Carroll [20]. Experiment and theoretical analysis have revealed that a mammalian brain not only displays in its storage of associative memories, but also modulates oscillatory neuronal synchronization by selective perceive attention [21,22]. The difference of using the benefits between synchronized stability and synchronized bifurcation is that memorized images correspond to equilibrium point attractors in the former and limit cycle attractors in the latter. In the theory and applications of content addressable memories, a stable solution can be used as coded information of a memory of the system to be stored and retrieved [7], pattern recognition by coupled neural networks consisting in convergence to the corresponding limit cycle attractor, which stores and retrieves complex oscillatory patterns in the synchronization states [23]. Periodic oscillation in neural networks is an interesting phenomenon, like many biological and cognitive activities [1]. So, how to understand the synchronized stability and synchronized bifurcation is very useful. In [24], Wei and Yuan considered the synchronized periodic oscillation in a ring neural network model with two different delays.

Reaction–diffusion (RD) mechanisms can describe many biological phenomena such as neuron firing in the brain, the heartbeat, cellular organization activities or even biological disorders such as fibrillation [37]. It is known that the foundations of neural processing refer to a phenomenon which takes place both in space and in time and involves an ensemble of neurons mutually connected, their dynamics is governed by the law of diffusion [35]. In signal transmission, the signal will become weak due to diffusion [36]. In addition, inspired by [25,26], we know that not only the evolution time of each variable and its position (space) but also the interactions deriving from the space-distributed structure of the whole networks determine the whole structure and dynamic behavior of multi-layer cellular neural networks seriously and intensively. Therefore, it is essential to consider the state variables that are varying not only with time but also with space [27–31] and reaction–diffusion effects cannot be neglected in both biological and man-made neural networks.

The simplest model to display features of neural interaction comprised of two coupled neural systems. Starting from this simplest network motif, larger networks can be built, and their effects may be studied. So, we focus on the simplest example in which each network copy is capable of oscillation, namely, a pair of simple loops of three neurons. With these in mind, based on the models in [12] and [14], we consider two kinds neural networks coupled by two sub-loop networks, each including three neurons: one way with delay in coupling; the other way with delay in sub-networks, both shown as follows (Fig. 1).

1) Coupled loops with delay

Consider a pair of loops with delayed coupling connection

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = d\nabla^2 u_1(t,x) + bf\bigl(u_2(t,x)\bigr) - u_1(t,x), \\ \dfrac{\partial u_2(t,x)}{\partial t} = d\nabla^2 u_2(t,x) + bf\bigl(u_3(t,x)\bigr) - u_2(t,x), \\ \dfrac{\partial u_3(t,x)}{\partial t} = d\nabla^2 u_3(t,x) + bf\bigl(u_1(t,x)\bigr) - u_3(t,x) \\ \qquad\qquad + cf\bigl(v_3(t-\tau,x)\bigr), \\ \dfrac{\partial v_1(t,x)}{\partial t} = d\nabla^2 v_1(t,x) + bf\bigl(v_2(t,x)\bigr) - v_1(t,x), \\ \dfrac{\partial v_2(t,x)}{\partial t} = d\nabla^2 v_2(t,x) + bf\bigl(v_3(t,x)\bigr) - v_2(t,x), \\ \dfrac{\partial v_3(t,x)}{\partial t} = d\nabla^2 v_3(t,x) + bf\bigl(v_1(t,x)\bigr) - v_3(t,x) \\ \qquad\qquad + cf\bigl(u_3(t-\tau,x)\bigr), \end{cases} \tag{1}$$

the Neumann boundary and initial conditions are given by

$$\begin{cases} \dfrac{\partial u_i}{\partial n} := \dfrac{\partial u_i}{\partial x} = 0, \\ \dfrac{\partial v_i}{\partial n} := \dfrac{\partial v_i}{\partial x} = 0, \\ t \ge 0, \quad x = 0, \pi, \quad i = 1,2,3. \end{cases} \tag{2}$$

$$\begin{cases} u_i(s,x) = \eta_i(s,x), \\ v_i(s,x) = \zeta_i(s,x), \\ (s,x) \in [-\tau, 0] \times [0,\pi], \quad i = 1,2,3. \end{cases} \tag{3}$$

2) Ring Structure with delay

Consider a pair of loops with delays in sub-network

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = d\nabla^2 u_1(t,x) + bf\bigl(u_2(t,x)\bigr) - u_1(t,x), \\ \dfrac{\partial u_2(t,x)}{\partial t} = d\nabla^2 u_2(t,x) + bf\bigl(u_3(t,x)\bigr) - u_2(t,x), \\ \dfrac{\partial u_3(t,x)}{\partial t} = d\nabla^2 u_3(t,x) + bf\bigl(u_1(t-\tau,x)\bigr) \\ \qquad\qquad - u_3(t,x) + cf\bigl(v_3(t,x)\bigr), \\ \dfrac{\partial v_1(t,x)}{\partial t} = d\nabla^2 v_1(t,x) + bf\bigl(v_2(t,x)\bigr) - v_1(t,x), \\ \dfrac{\partial v_2(t,x)}{\partial t} = d\nabla^2 v_2(t,x) + bf\bigl(v_3(t,x)\bigr) - v_2(t,x), \\ \dfrac{\partial v_3(t,x)}{\partial t} = d\nabla^2 v_3(t,x) + bf\bigl(v_1(t-\tau,x)\bigr) \\ \qquad\qquad - v_3(t,x) + cf\bigl(u_3(t,x)\bigr), \end{cases} \tag{4}$$

with Neumann boundary conditions (2), initial conditions are given by