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Modulational instability in wind-forced waves

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ABSTRACT

We consider the wind-forced nonlinear Schrödinger (NLS) equation obtained in the potential flow framework when the Miles growth rate is of the order of the wave steepness. In this case, the form of the wind-forcing terms gives rise to the enhancement of the modulational instability and to a band of positive gain with infinite width. This regime is characterised by the fact that the ratio between wave momentum and norm is not a constant of motion, in contrast to what happens in the standard case where the Miles growth rate is of the order of the steepness squared.

1. Introduction

The modulational instability (known as Benjamin–Feir instability in the context of fluid dynamics [1,2]) is ubiquitous in physics, it occurs in nonlinear waves within numerous physical situations (water waves, plasma waves, laser beams, electromagnetic transmission lines, ...) [3,4] and it is one of the possible mechanisms of catastrophic growth and generation of rogue waves in the ocean [5].

The stability properties of the wavetrains rely on the form of the damping/pumping terms in the governing equations which, in the context of water waves, depend on the wind providing energy to the system [6–8]. Modelling the effects of wind on ocean waves is a very complex task due to turbulence in both the atmospheric and the oceanic boundary layers, and nonlinearities in the propagation of the gravity waves at the interface. The problem has been simplified by assuming quasi-laminar airflows [9] through the Miles mechanism [10], quasi-linear theory in wind-wave generation (the Janssen mechanism [11]) and different approximations in the wave dynamics (i.e. in the Navier–Stokes equations or the Euler equations) to obtain mathematical models for the propagation of surface gravity waves which can be handled analytically. The wind can induce either damping or forcing terms in the resulting equations [12,8] depending on its speed and direction relative to the wave propagation. Many experiments have been performed to investigate how surface waves and modulational instability are affected by wind and dissipation [13–17], sometimes with contrasting results regarding in particular the values of the damping rates induced by winds blowing slower or opposite to the wave velocity [18–20].

The effect of wind can be modelled in the framework of the Miles mechanism [10] and the potential flow approximation [21] for deep-water waves. The growth rate Γ_M/f of the wave energy (normalised with respect to the frequency of the carrier wave) is most often taken of the same order as the dissipation, hence at the $\Gamma_M/f = O(\epsilon^2)$, and the resulting envelope equation at third-order in the wave steepness ϵ is given by a wind-forced nonlinear Schrödinger (NLS) equation [12,8,22] of the form

$$i\frac{\partial a}{\partial t} - \beta_1\frac{\partial^2 a}{\partial x^2} - M|a|^2 a = i\left(\frac{\Gamma_M}{2} - 2\nu k^2\right)a \tag{1}$$

where $\beta_1 = -(dc_g/dk)/2 = \omega/(8k^2)$, $M = \omega k^2/2$, and ν is the kinematic viscosity.

Recently we have derived the wind-forced NLS for stronger wind forcing, with a growth rate Γ_M/f of the wave energy of the same order as the steepness [23], $\Gamma_M/f = O(\epsilon)$. In this case, the envelope equation obtained by the multiple-scale perturbation method at third-order in ϵ reads

$$i\frac{\partial a}{\partial t} - \beta_1\frac{\partial^2 a}{\partial x^2} - Ma|a|^2 = \left(\beta_2\frac{\partial}{\partial x} + \beta_3 - 2i\nu k^2\right)a \tag{2}$$

where $\beta_2 = 3\Gamma_M/(4k)$ and $\beta_3 = \Gamma_M^2/(8\omega)$. As compared to Eq. (1), the latter equation contains two additional forcing terms, namely the terms proportional to β_2 and β_3.

In this Letter, we investigate the effects of the wind-forcing terms in Eq. (2) on the modulational instability (Section 2) and

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compare it to the well-known case described by Eq. (1) for reference. We show that considering the wave-energy growth rate at the first order in steepness results in widely extending the spectral range of the modulational instability gain. Besides, we show (Section 3) that the way wind-forcing is considered affects the ratio of the momentum to the norm of the pulse, that is conserved only if the growth rate is limited to the second order in steepness. We compare this finding with recent sets of experiments where either the carrier wave amplitude or the initial perturbation amplitudes are sufficiently large and the modulational instability is enhanced [15], suggesting the physical relevance of considering the model given by Eq. (2). We discuss the main results in Section 4 and we draw the conclusions in Section 5.

2. Modulational instability

Benjamin and Feir [1] showed that inviscid deep-water wavetrains are unstable to small perturbations of other waves travelling in the same direction with frequencies within the band of positive gain. We compare here the modulation instability when wind forcing terms are included in the envelope equations in two different regimes: low Miles growth rates $\Gamma_M/f = O(\epsilon^2)$ (that is the well-known standard case that we develop for reference) and high Miles growth rates $\Gamma_M/f = O(\epsilon)$.

2.1. Low growth rates

We review here for reference the standard case where the envelope equation is given by Eq. (1). This will be useful to set-up the formalism and to compare with results obtained when considering growth rates at the first order in steepness.

By defining $\tau = \omega t$, $\xi = 2kx$, $\Gamma = \Gamma_M/(2\omega)$, $\delta = 2\nu k^2/\omega$, $K = \Gamma - \delta$, and $A = ka/\sqrt{2}$, Eq. (1) reduces to [8]

$$iA_\tau - \frac{1}{2}A_{\xi\xi} - A|A|^2 = iKA \tag{3}$$

The factor K on the right-hand side can be positive, null or negative depending on the relative importance of the viscosity term δ with respect to the wind-forcing term Γ. The Stokes-like wave, which is a solution of Eq. (3) independent on ξ, is given by

$$A_S(\tau) = A_0 e^{K\tau} e^{-ib(\tau)}, \qquad b(\tau) = \frac{|A_0|^2}{2K}\left(e^{2K\tau} - 1\right) \tag{4}$$

Note that for $K = 0$, we get $b(\tau) = |A_0|^2\tau$, which is valid in the inviscid case. Following previous studies [15,12,8], the Stokes-like wave is perturbed as follows

$$A(\xi,\tau) = A_S(\tau)\left[1 + \delta_0\zeta(\xi,\tau)\right] \tag{5}$$

with δ_0 infinitesimal and $\zeta(\xi,\tau) = M(\xi,\tau) + iN(\xi,\tau)$. Substituting into Eq. (3) gives the following system of equations

$$M_\tau - \frac{1}{2}N_{\xi\xi} = 0 \tag{6}$$

$$N_\tau + \frac{1}{2}M_{\xi\xi} + 2|A_S|^2 M = 0 \tag{7}$$

By choosing perturbations of the form

$$M(\xi,\tau) = \Re\left\{M_0(\tau)e^{i\ell\xi}\right\} \tag{8}$$

$$N(\xi,\tau) = \Re\left\{N_0(\tau)e^{i\ell\xi}\right\} \tag{9}$$

where ℓ is the modulational wavenumber, the previous system becomes

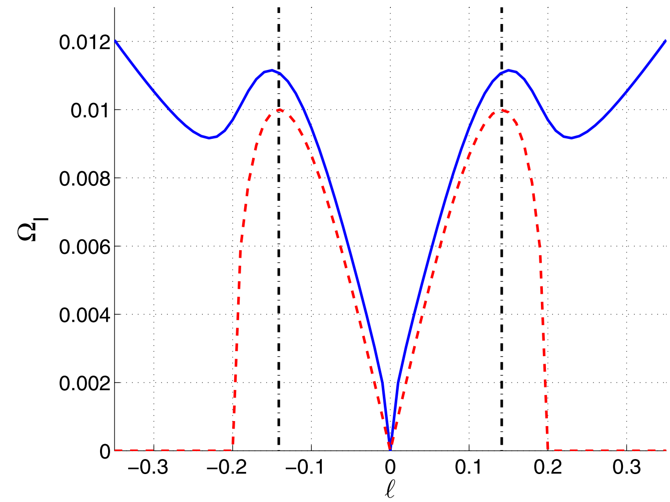

Fig. 1. Band of positive gain for the modulational wavenumbers ℓ with $|A_S| = 0.1$: low Miles growth-rates (dashed red line) and high Miles growth-rates (solid blue line) with $\Gamma_M/f = \epsilon = \sqrt{2}|A_S|$. Dash-dotted vertical lines correspond to $\ell^* = \pm\sqrt{2}|A_S|$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\frac{dM_0}{d\tau} + \frac{\ell^2}{2}N_0 = 0 \tag{10}$$

$$\frac{dN_0}{d\tau} - \left(\frac{\ell^2}{2} - 2|A_S|^2\right)M_0 = 0 \tag{11}$$

which corresponds to the following equation [15,12,8]

$$\frac{d^2M_0}{d\tau^2} + \frac{\ell^2}{2}\left(\frac{\ell^2}{2} - 2|A_0|^2 e^{2K\tau}\right)M_0 = 0 \tag{12}$$

In the case $K = 0$, this differential equation has constant coefficients and by setting $M_0(\tau) = \tilde{M}e^{-i\Omega\tau}$, one gets the dispersion relation [15]

$$\Omega = \pm\frac{\ell}{\sqrt{2}}\sqrt{\frac{\ell^2}{2} - 2|A_0|^2} \tag{13}$$

In the case $K \neq 0$, Eq. (12) is a Sturm–Liouville problem [15] which must be analysed as in [15,8]. The presence of oscillatory or exponentially growing solutions depends on the sign of the factor $(\ell^2/2 - 2|A_0|^2 e^{2K\tau})$ in Eq. (12). Growing perturbations of the Stokes-like solution appear in a limited range of modulational wavenumbers [15]

$$\ell^2 < 4|A_0|^2 e^{2K\tau} \tag{14}$$

The stability range expands (contracts) with time in the presence of pumping $K = \Gamma - \delta > 0$ (damping $\Gamma < \delta$), but the Benjamin–Feir instability gain is independent from the pumping/damping term [15,7]. In other words, the dependence on Γ is only within the exponential term which appears in the Stokes wave amplitude $|A_S|$ and determines expansion or contraction depending on the sign of K. The range where modulational wavenumbers become unstable is shown in Fig. 1, dashed line, for $|A_S| = 0.1$. The maximum growth rate occurs at $\ell^* = \pm\sqrt{2}|A_S|$ (see vertical dash-dotted lines in Fig. 1)

$$\Omega_I(\ell = \ell^*) = |A_0|^2 e^{2K\tau} = |A_S|^2 \tag{15}$$

2.2. High growth rates

Here we conduct a similar procedure in the case of the envelope equation (2) obtained from the full nonlinear gravity-wave