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Continuous symmetries of certain nonlinear partial difference equations and their reductions

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ABSTRACT

In this article, Quispel, Roberts and Thompson type of nonlinear partial difference equation with two independent variables is considered and identified five distinct nonlinear partial difference equations admitting continuous point symmetries quadratic in the dependent variable. Using the degree growth of iterates the integrability nature of the obtained nonlinear partial difference equations is discussed. It is also shown how to derive higher order ordinary difference equations from the periodic reduction of the identified nonlinear partial difference equations. The integrability nature of the obtained ordinary difference equations is investigated wherever possible.

1. Introduction

Lie groups and Lie algebras are mathematical objects which have originated in the seminal work of Sophus Lie (1842–1899) on solving differential equations by quadrature, using symmetry methods. Lie symmetries approach originally introduced by Sophus Lie has been used as a tool to unify various integration techniques for ordinary differential equations (ODEs) and has played a very important role in the study of both ODEs and partial differential equations (PDEs). Symmetry groups are invariant transformations which do not alter the structural form of the equation under investigation. Once the symmetry group of a system of differential equations is known, it can be used to generate new solutions from the old ones, often interesting ones from trivial ones [20]. It can be used to classify solutions into conjugacy classes and to classify and simplify differential equations. An important application of the symmetry approach is the reduction of an ODE to a lower order one, the reduction of a PDE to one with fewer independent variables. The usefulness of Lie symmetry approach has been widely illustrated for a variety of dynamical systems governed by both nonlinear ODEs and PDEs which arise in different contexts [3,8,9,19,20,27] during the past several decades [3,8,9,19,20,27]. During 1980s Maeda [15–18] has extended Lie symmetries approach to discrete systems governed by nonlinear mappings or ordinary difference equations (O∆Es) and demonstrated how it provides an effective tool to derive their continuous point symmetries. Later on the Lie symmetry approach has been further developed by Levi and Winternitz [10,12], Quispel et al. [22], Levi et al. [11,13] and others [4,7,23,24] to nonlinear discrete systems governed by partial differential–difference equations (PD∆Es) and O∆Es. Also, several groups have profitably exploited to derive mathematical structures related with integrability of different nonlinear P∆Es possessing solitons [1,6,25]. Though the Lie symmetry approach has been extended to discrete nonlinear systems governed by lattice equations or nonlinear partial difference equations (P∆∆Es), its effectiveness has not yet been demonstrated widely. The objective of this article is to illustrate its usefulness on other nonlinear P∆∆Es. More specifically a scalar nonlinear P∆∆E of Quispel, Roberts and Thompson (QRT) type [21]

$$v(l+1,m+1) = \frac{f_1(v(l,m+1),v(l+1,m)) - v(l,m) f_2(v(l,m+1),v(l+1,m))}{f_3(v(l,m+1),v(l+1,m)) - v(l,m) f_4(v(l,m+1),v(l+1,m))}$$

is considered and under what conditions on f_i it possesses continuous point symmetries quadratic in the dependent variable is investigated. The integrability nature of obtained P∆∆Es is analyzed using the degree growth of iterates [26], another characteristic of integrable discrete systems. Also, it is shown how higher order O∆Es are explicitly derived.

The paper is organized as follows. In Section 2, QRT type of nonlinear P∆∆E is considered as mentioned above and under what conditions on f_i it possesses continuous point symmetries

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quadratic in the dependent variable is investigated. In Section 3, the integrability nature of the obtained P$\Delta\Delta$Es is analyzed through the degree growth of iterates. In Section 4, higher order OΔEs are derived from the periodic reduction of the obtained P$\Delta\Delta$Es. Also, it is shown that the derived OΔEs are measure preserving and they admit sufficient number of integrals, if exist, leading to their integrability. In Section 5, a brief summary of the obtained results and concluding remarks are provided.

2. Lie point symmetries of partial difference equations

Consider a scalar nonlinear PΔE of the form

$$v(l+1,m+1)=F\big(v(l,m),v(l,m+1),v(l+1,m)\big), \tag{2.1}$$

where $F(v(l,m),v(l,m+1),v(l+1,m))$ is an arbitrary function. Let us assume that (2.1) is invariant under a one-parameter (ϵ) continuous point transformations

$$l^*=l,\qquad m^*=m, \tag{2.2}$$

$$v^*=v(l,m)+\epsilon\eta\big(l,m,v(l,m)\big)+O\left(\epsilon^2\right) \tag{2.3}$$

with infinitesimal generator

$$X=\eta\big(l,m,v(l,m)\big)\frac{\partial}{\partial v(l,m)} \tag{2.4}$$

provided any solution $v(l,m)$ satisfies (2.1). Hereafter denoting

$$F\equiv F\big(v(l,m),v(l,m+1),v(l+1,m)\big),\qquad v_m^l\equiv v(l,m),$$
$$v_{m+1}^l\equiv v(l,m+1),\qquad v_m^{l+1}\equiv v(l+1,m),$$
$$v_{m+1}^{l+1}\equiv v(l+1,m+1)$$

unless otherwise specified. Then the invariant equation reads

$$\begin{aligned}\eta(l+1,m+1,F)={}&\eta\big(l,m+1,v_{m+1}^l\big)\frac{\partial F}{\partial v_{m+1}^l}\\&+\eta\big(l+1,m,v_m^{l+1}\big)\frac{\partial F}{\partial v_m^{l+1}}\\&+\eta\big(l,m,v_m^l\big)\frac{\partial F}{\partial v_m^l}.\end{aligned} \tag{2.5}$$

Eq. (2.5) is a functional difference equation and there is no known method to solve it. In the present work, the investigation is restricted to QRT type nonlinear P$\Delta\Delta$E having the form

$$F=\frac{f_1-v_m^lf_2}{f_3-v_m^lf_4}, \tag{2.6}$$

where f_1, f_2, f_3 and f_4 are arbitrary functions of v_{m+1}^l and v_m^{l+1}. In order to solve the invariance equation (2.5) with (2.6), assume that

$$\eta\big(l,m,v_m^l\big)=A(l,m)\big(c_1+c_2v_m^l+c_3\big(v_m^l\big)^2\big), \tag{2.7}$$

where $A(l,m)$ is an arbitrary function and c_i, $i=1,2,3$ are arbitrary parameters. To start with let $A(l,m)=1$. Using (2.7) along with their shifts in Eq. (2.5) and equating powers of $(v_m^l)^j$, $j=0,1,2$ to zero the following three equations are obtained:

$$\begin{aligned}&f_2^2\big(c_1+c_2v_{m+1}^l+c_3\big(v_{m+1}^l\big)^2\big)\frac{\partial}{\partial v_{m+1}^l}\left(\frac{f_4}{f_2}\right)\\&\quad+f_2^2\big(c_1+c_2v_m^{l+1}+c_3\big(v_m^{l+1}\big)^2\big)\frac{\partial}{\partial v_m^{l+1}}\left(\frac{f_4}{f_2}\right)\\&\quad+\big[c_1f_4^2+c_2f_2f_4+c_3\big(f_2^2+f_2f_3-f_1f_4\big)\big]=0,\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\big(c_1+c_2v_{m+1}^l+c_3\big(v_{m+1}^l\big)^2\big)\\&\quad\times\left[f_1^2\frac{\partial}{\partial v_{m+1}^l}\left(\frac{f_4}{f_1}\right)+f_2^2\frac{\partial}{\partial v_{m+1}^l}\left(\frac{f_3}{f_2}\right)\right]\\&\quad+\big(c_1+c_2v_m^{l+1}+c_3\big(v_m^{l+1}\big)^2\big)\\&\quad\times\left[f_1^2\frac{\partial}{\partial v_m^{l+1}}\left(\frac{f_4}{f_1}\right)+f_2^2\frac{\partial}{\partial v_m^{l+1}}\left(\frac{f_3}{f_2}\right)\right]\\&\quad+2(c_1f_3f_4+c_2f_1f_4+c_3f_1f_2)=0,\end{aligned} \tag{2.9}$$

$$\begin{aligned}&f_1^2\big(c_1+c_2v_{m+1}^l+c_3\big(v_{m+1}^l\big)^2\big)\frac{\partial}{\partial v_{m+1}^l}\left(\frac{f_3}{f_1}\right)\\&\quad+f_1^2\big(c_1+c_2v_m^{l+1}+c_3\big(v_m^{l+1}\big)^2\big)\frac{\partial}{\partial v_m^{l+1}}\left(\frac{f_3}{f_1}\right)\\&\quad+\big[c_1\big(f_3^2+f_2f_3-f_1f_4\big)+c_2f_1f_3+c_3f_1^2\big]=0.\end{aligned} \tag{2.10}$$

Then there exist different possibilities which will be discussed below separately.

Case 1: $c_1\neq0$, $c_2\neq0$, $c_3\neq0$

After a detailed calculation it is found that Eqs. (2.8)–(2.10) satisfy identically provided

$$\begin{aligned}&f_1=1,\qquad f_2=\frac{a}{v_{m+1}^l}+\frac{(1-a)}{v_m^{l+1}},\\&f_3=\frac{a}{v_m^{l+1}}+\frac{(1-a)}{v_{m+1}^l},\qquad f_4=\frac{1}{v_{m+1}^lv_m^{l+1}},\end{aligned} \tag{2.11}$$

where a is an arbitrary parameter and so the QRT P$\Delta\Delta$Es (2.6) becomes

$$v_{m+1}^{l+1}=\frac{v_{m+1}^lv_m^{l+1}-v_m^l\big[av_m^{l+1}+(1-a)v_{m+1}^l\big]}{\big[av_{m+1}^l+(1-a)v_m^{l+1}\big]-v_m^l} \tag{2.12}$$

which is invariant under

$$l^*=l,\qquad m^*=m,$$
$$v^*=v_m^l+\epsilon\big[c_1+c_2v_m^l+c_3\big(v_m^l\big)^2\big]+O\left(\epsilon^2\right)$$

with infinitesimal generator

$$X=\big[c_1+c_2v_m^l+c_3\big(v_m^l\big)^2\big]\frac{\partial}{\partial v_m^l}$$

leading to the following generators:

$$X_1=\frac{\partial}{\partial v_m^l},\qquad X_2=v_m^l\frac{\partial}{\partial v_m^l},\qquad X_3=\big(v_m^l\big)^2\frac{\partial}{\partial v_m^l}.$$

It is straight forward to check that the above generators satisfy

$$[X_1,X_2]=X_1,\qquad [X_1,X_3]=2X_2,\qquad [X_2,X_3]=X_3$$

indicating that the underlying Lie algebra of P$\Delta\Delta$E (2.12) is not solvable [3].

Case 2: $c_1=0$, $c_2\neq0$, $c_3\neq0$

Proceeding as before, we find that Eqs. (2.8)–(2.10) satisfy provided

$$\begin{aligned}&f_1=1,\qquad f_2=\frac{a}{v_{m+1}^l}+\frac{(1-a)}{v_m^{l+1}},\\&f_3=\frac{a}{v_m^{l+1}}+\frac{(1-a)}{v_{m+1}^l},\qquad f_4=\frac{1}{v_{m+1}^lv_m^{l+1}}+b\left(\frac{v_m^{l+1}-v_{m+1}^l}{v_{m+1}^lv_m^{l+1}}\right)^2\end{aligned} \tag{2.13}$$