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Special polynomials associated with the fourth order analogue to the Painlevé equations

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Abstract

Rational solutions of the fourth order analogue to the Painlevé equations are classified. Special polynomials associated with the rational solutions are introduced. The structure of the polynomials is found. Formulae for their coefficients and degrees are derived. It is shown that special solutions of the Fordy–Gibbons, the Caudrey–Dodd–Gibbon and the Kaup–Kupershmidt equations can be expressed through solutions of the equation studied.
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1. Introduction

It is well known that the general solutions of the six Painlevé equations (P_1–P_6) cannot be expressed through known elementary or special functions, in other words they define new transcendental functions. However, the equations P_2–P_6 possess hierarchies of rational and algebraic solutions at certain values of the parameters. It turned out that such solutions could be described using families of special polynomials. The history of this question is as follows.

Yablonskii and Vorob'ev were first who expressed the rational solutions of P_2 via the logarithmic derivative of the polynomials, which now go under the name of the Yablonskii–Vorob'ev polynomials [1,2]. Later Okamoto suggested special polynomials for certain rational solutions of P_4 [3]. Umemura derived analogous polynomials for some rational and algebraic solutions of P_3 and P_5 [4]. All these polynomials possess a number of interesting properties. For example, they can be expressed in terms of Schur polynomials. Besides that the polynomials arise as the so-called tau-functions and satisfy recurrence relations of Toda type. Recently these polynomials have been intensively studied [5–10].

Apart from rational and algebraic solutions P_2–P_6 possess one-parameter families of solutions expressible in terms of the classical special functions. In particular cases these special function solutions are reduced to classical orthogonal polynomials and, consequently give rational solutions of the Painlevé equation in question. For P_4 these polynomials are Hermite polynomials, for P_3 and P_5 Laguerre polynomials and for P_6 Jacobi polynomials.

Not long ago Clarkson and Mansfield suggested special polynomials for the equations of the P_2 hierarchy [11]. Also they studied the location of their roots in the complex plane and showed that the roots have a very regular structure.

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The aim of this work is to introduce special polynomials related to rational solutions of the following analogue to the Painlevé equations

$$w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5w w_z^2 + w^5 - zw - \beta = 0. \tag{1.1}$$

Originally this equation was found from the Fordy–Gibbons equation [12]

$$\omega_t + \omega_{xxxxx} + 5\omega_x \omega_{xxx} - 5\omega^2 \omega_{xxx} + 5\omega_{xx}^2 - 20\omega\omega_x\omega_{xx} - 5\omega_x^3 + 5\omega^4\omega_x = 0 \tag{1.2}$$

through the scaling reduction

$$\omega(x,t) = (5t)^{-1/5} w(z), \quad z = x(5t)^{-1/5}. \tag{1.3}$$

Eq. (1.1) was first considered in [13] and later in works [14–20]. This equation has a number of properties similar to those of the Painlevé equations. More exactly it possesses the Bäcklund transformations, the Lax pair, rational and special solutions at certain values of the parameter β [13,14]. These special solutions are expressible in terms of the first Painlevé transcendent [15,18]. The Cauchy problem for this equation can be solved by the isomonodromic deformation method. Apparently the equation (1.1) defines new transcendental functions like the Painlevé equations do.

Let us demonstrate that special solutions of the Caudrey–Dodd–Gibbon equation can be expressed through solutions of (1.1).

The Caudrey–Dodd–Gibbon equation (Savada–Kotera equation) can be written as [21–23]

$$u_t + u_{xxxxx} - 5u u_{xxx} - 5u_x u_{xx} + 5u^2 u_x = 0. \tag{1.4}$$

This equation has the self-similar reduction

$$u(x,t) = (5t)^{-2/5} y(z), \quad z = x(5t)^{-1/5}, \tag{1.5}$$

where $y(z)$ satisfies the equation

$$y_{zzzzz} - 5y y_{zzz} - 5y_z y_{zz} + 5y^2 y_z - 2y - z y_z = 0. \tag{1.6}$$

The Miura transformation $y(z) = w_z + w^2$ relates solutions of (1.6) to solutions of the equation

$$\left(\frac{d}{dz} + 2w\right)\frac{d}{dz}\left(w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5w w_z^2 + w^5 - wz - \beta\right) = 0. \tag{1.7}$$

Thus we see that for any solution of (1.1) there exists a solution of (1.4).

The Kaup–Kupershmidt equation [23,24]

$$v_t + v_{xxxxx} + 10v v_{xxx} + 25v_x v_{xx} + 20v^2 v_x = 0 \tag{1.8}$$

also possesses solutions, which can be expressed via solutions of (1.1). Indeed it has the self-similar reduction

$$v(x,t) = (5t)^{-2/5} y(z), \quad z = x(5t)^{-1/5} \tag{1.9}$$

with $y(z)$ satisfying the equation

$$y_{zzzzz} + 10y y_{zzz} + 25y_z y_{zz} + 20y^2 y_z - z y_z - 2y = 0. \tag{1.10}$$

After making the Miura transformation $y(z) = w_z - \frac{1}{2}w^2$ we obtain

$$\left(\frac{d}{dz} - w\right)\frac{d}{dz}\left(w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5w w_z^2 + w^5 - wz - \beta\right) = 0. \tag{1.11}$$

Hence the Fordy–Gibbons equation (1.2), the Caudrey–Dodd–Gibbon equation (1.4), and the Kaup–Kupershmidt equation (1.8) admit solutions in terms of solutions of (1.1).

This Letter is organized as follows. In Section 2 we classify rational solutions of (1.1) and construct associated polynomials. In Section 3 we find some correlations for the poles of rational solutions.

2. Special polynomials associated with rational solutions of Eq. (1.1)

Let us briefly review some facts concerning Eq. (1.1) needed later. Suppose $w \equiv w(z;\beta)$ is a solution of (1.1). Then the transformations

$$\begin{aligned} T_{2-\beta}: &\quad w(z;2-\beta) = w + \frac{2\beta - 2}{z - w_{zzz} + w w_{zz} - 3w_z^2 + 4w^2 w_z - w^4}, \\ T_{-1-\beta}: &\quad w(z;-1-\beta) = w + \frac{2\beta + 1}{z + 2w_{zzz} + 4w w_{zz} + 3w_z^2 - 2w^2 w_z - w^4}, \end{aligned} \tag{2.1}$$

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