Physics Letters A 378 (2014) 2931–2937

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

$SU(1,1)$ coherent states for Dirac–Kepler–Coulomb problem in $D+1$ dimensions with scalar and vector potentials

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ARTICLE INFO

Article history:
Received 12 November 2013
Received in revised form 26 March 2014
Accepted 11 August 2014
Available online 23 August 2014
Communicated by A.P. Fordy

Keywords:
Lie algebras
Coherent states
Dirac equation
Scalar–vector potentials
Higher dimensions

ABSTRACT

We decouple the Dirac's radial equations in $D+1$ dimensions with Coulomb-type scalar and vector potentials through appropriate transformations. We study each of these uncoupled second-order equations in an algebraic way by using an $su(1,1)$ algebra realization. Based on the theory of irreducible representations, we find the energy spectrum and the radial eigenfunctions. We construct the Perelomov coherent states for the Sturmian basis, which is the basis for the unitary irreducible representation of the $su(1,1)$ Lie algebra. The physical radial coherent states for our problem are obtained by applying the inverse original transformations to the Sturmian coherent states.

1. Introduction

Since Schrödinger introduced the harmonic oscillator coherent states [1], they have played a fundamental role in quantum mechanics. These coherent states are related to the Heisenberg–Weyl group. The works of Barut [2] and Perelomov [3] generalized the harmonic oscillator coherent states to those of any algebra of a symmetry group.

The coherent states have been obtained successfully for many problems, reported in the references [4–8]. Related to the Perelomov coherent states for the $su(2)$ and $su(1,1)$ Lie algebras, several works have been published, some of them are [9–11].

As one of the few exactly solvable problems in physics, the Kepler–Coulomb problem has been treated in several ways, analytical [12–15], factorization methods [16,17], shape-invariance [18], SUSY QM for the first [19] and second-order [20] differential equations, two-variable realizations of the $su(2)$ Lie algebra [21] and using the Biedenharn–Temple operator [22]. Its solubility is due to the conservation of the total angular momentum, and the Dirac and Johnson–Lippmann operators [23]. In fact, is has been shown that the supersymmetry charges are generated by the Johnson–Lippmann operator [23]. Joseph was the first in studying the Kepler–Coulomb problem in $D+1$ dimensions by means of self-adjoint operators [24]. The energy spectrum and the eigenfunctions of this problem were obtained by solving the confluent hypergeometric equation [25,26]. Moreover, in [27] the Johnson–Lippmann operator for this potential has been constructed and used to generate the SUSY charges.

The Dirac equation with Coulomb-type vector and scalar potentials in $3+1$ dimensions has been solved by using SUSY QM [28] and the matrix form of SUSY QM based on intertwining operators [29]. For the $(D+1)$-dimensional case it was treated by reducing the uncoupled radial second-order equations to those of the confluent hypergeometric functions [25,26]. In recent works, it has been studied the Dirac equation for the three-dimensional Kepler–Coulomb problem [30], and with Coulomb-type scalar and vector potentials in $D+1$ dimensions from an $su(1,1)$ algebraic approach [31]. Also, a Johnson–Lippmann operator has been constructed for Coulomb-type scalar and vector potential in general spatial dimensions. It was used to generate the SUSY charges [32].

In the relativistic regimen the spectrum of the Klein–Gordon Coulomb problem was calculated by using the $SO(2,1)$ coherent-state theory [33]. For the Dirac problem, only the coherent states for the three-dimensional relativistic Kepler–Coulomb potential have been

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http://dx.doi.org/10.1016/j.physleta.2014.08.023

treated [34]. This study was based on the fact that the uncoupled second-order differential equations admits a two-variable $SU(2)$ symmetry. The additional variable to the radial coordinate was needed in order to close the $su(2)$ Lie algebra [21]. We notice that in references [33,34] the explicit closed form of the relativistic coherent states have not been obtained. However, there are no works on relativistic coherent states in which scalar and vector potentials are considered, in three or higher spatial dimensions.

The aim of the present work is to construct the radial $SU(1,1)$ Perelomov coherent states for the relativistic Kepler–Coulomb problem in $D+1$ dimensions with Coulomb-type scalar and vector potentials. Our treatment is restricted for bound states. We decouple the first-order Dirac equations to obtain the second-order differential equations. Each of these equations is written in terms of a set of radial operators which close the $su(1,1)$ Lie algebra. One of these operators is the so-called scaling operator. By an appropriate choice of the scaling parameter we diagonalize the second-order radial equation. We use the theory of unitary representations to find the energy spectrum and the radial wave functions from the Sturmian basis (group basis). We construct the $SU(1,1)$ Perelomov coherent states for the Sturmian basis and the inverse transformations are applied to these states to obtain the radial $SU(1,1)$ Perelomov coherent states for the relativistic Kepler–Coulomb problem.

This work is organized as follows. In Section 2 we obtain the uncoupled second-order differential equations satisfied by the radial components. The $su(1,1)$ Lie algebra generators for the uncoupled second-order differential equations are introduced. The energy spectrum and radial wave functions are found. In Section 3, we obtain the explicit expression of $SU(1,1)$ Perelomov coherent states for the relativistic Kepler–Coulomb problem in $D+1$ dimensions with scalar and vector potentials. Finally, we give some concluding remarks.

2. Second order radial equations

The Dirac equation in $D+1$ dimensions for a central field is given by [26]

$$i\frac{\partial\Psi}{\partial t}=H\Psi,\qquad H=\sum_{a=1}^{D}\alpha_a p_a+\beta\big(m+V_s(r)\big)+V_v(r),\tag{1}$$

with $\hbar=c=1$, m is the mass of the particle, V_s and V_v are the spherically symmetric scalar and vector potentials, respectively and

$$p_a=-i\partial_a=-i\frac{\partial}{\partial x_a}\quad 1\le a\le D.\tag{2}$$

In (1), α_a and β satisfy the anticommutation relations

$$\begin{aligned}&\alpha_a\alpha_b+\alpha_b\alpha_a=2\delta_{ab}\mathbf{1},\\&\alpha_a\beta+\beta\alpha_a=0,\\&\alpha_a^2=\beta^2=1.\end{aligned}\tag{3}$$

In D spatial dimensions, the orbital angular momentum operators L_{ab}, the spinor operators S_{ab} and the total angular momentum operators J_{ab} are defined as

$$\begin{aligned}&L_{ab}=-L_{ba}=ix_a\frac{\partial}{\partial x_b}-ix_b\frac{\partial}{\partial x_a},\qquad S_{ab}=-S_{ba}=i\frac{\alpha_a\alpha_b}{2},\qquad J_{ab}=L_{ab}+S_{ab}.\\&L^2=\sum_{a<b}^{D}L_{ab}^2,\qquad S^2=\sum_{a<b}^{D}S_{ab}^2,\qquad J^2=\sum_{a<b}^{D}J_{ab}^2,\quad 1\le a\le b\le D.\end{aligned}\tag{4}$$

Hence, for a spherically symmetric potential, the total angular momentum operator J_{ab} and the spin–orbit operator $K_D=-\beta(J^2-L^2-S^2+\frac{(D-1)}{2})$ commute with the Dirac Hamiltonian. For a given total angular momentum j, the eigenvalues of the operator K_D are $\kappa_D=\pm(j+(D-2)/2)$, where the minus sign is for aligned spin $j=\ell+\frac{1}{2}$, and the plus sign is for unaligned spin $j=\ell-\frac{1}{2}$.

We propose the Dirac wave function of Eq. (1) to be of the form

$$\Psi(\vec{r},t)=r^{-\frac{D-1}{2}}\begin{pmatrix}F_{\kappa_D}(r)Y_{jm}^{\ell}(\Omega_D)\\ iG_{\kappa_D}(r)Y_{jm}^{\ell'}(\Omega_D)\end{pmatrix}e^{-iEt},\tag{5}$$

being $F_{k_D}(r)$ and $G_{k_D}(r)$ the radial functions, $Y_{jm}^{\ell}(\Omega_D)$ and $Y_{jm}^{\ell'}(\Omega_D)$ the hyperspherical harmonic functions coupled with the total angular momentum quantum number j, and E the energy. Thus, Eq. (1) leads to the radial equations

$$\begin{pmatrix}\frac{dF_{\kappa_D}}{dr}\\ \frac{dG_{\kappa_D}}{dr}\end{pmatrix}=\begin{pmatrix}-\frac{\kappa_D}{r} & V_s-V_v+m+E\\ V_v+V_s+m-E & \frac{\kappa_D}{r}\end{pmatrix}\begin{pmatrix}F_{\kappa_D}\\ G_{\kappa_D}\end{pmatrix}.\tag{6}$$

We consider the Coulomb-type scalar and vector potentials

$$V_v=-\frac{\alpha_v}{r},\qquad V_s=-\frac{\alpha_s}{r},\tag{7}$$

with α_v and α_s positive constants. Therefore, from Eq. (6) we obtain

$$\begin{pmatrix}\frac{dF_{k_D}}{dr}\\ \frac{dG_{k_D}}{dr}\end{pmatrix}+\frac{1}{r}\begin{pmatrix}k_D & \alpha_v-\alpha_s\\ \alpha_v+\alpha_s & -k_D\end{pmatrix}\begin{pmatrix}F_{k_D}\\ G_{k_D}\end{pmatrix}=\begin{pmatrix}0 & m+E\\ m-E & 0\end{pmatrix}\begin{pmatrix}F_{k_D}\\ G_{k_D}\end{pmatrix}.\tag{8}$$