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Positive stable preconditioners for symmetric indefinite linear systems arising from Helmholtz equations

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ABSTRACT

Using the finite difference method to discretize Helmholtz equations usually leads to a large spare linear system of equations $\mathcal{A}x = b$. Since the coefficient matrix $\mathcal{A}$ is frequently indefinite, it is difficult to solve iteratively. The approach taken in this Letter is to precondition this linear system with positive stable preconditioners and then to solve it iteratively using Krylov subspace methods. Numerical experiments are given in order to demonstrate the efficiency of the presented preconditioners.

1. Introduction

In computational electromagnetics and seismology, the finite difference method is one of the most effective and popular techniques. The finite difference method has many important applications such as time-harmonic wave propagations, scattering phenomena arising in acoustic and optical problems. More information about applications of this method in electromagnetics can be found in [1–3].

In this Letter, we are mainly interested in the following form of the Helmholtz equation:

$$\begin{cases} -\Delta u - pu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. Ω is a bounded region in $\mathbb{R}^2$. $p \geqslant 0$ is a real continuous coefficient function on $\bar{\Omega}$, while f and g are given continuous functions on $\bar{\Omega}$ and $\partial\Omega$, respectively.

To conveniently find numerical solutions of (1.1), the equation is discretized with the second-order difference scheme, in x-direction:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}) + \mathcal{O}(h^2),$$

and similarly in y-direction with a constant mesh spacing h in both directions. This approach yields the following linear system:

$$\mathcal{A}x = (A - h^2 D)x = b, \tag{1.2}$$

where A is the symmetric positive definite M-matrix arising from the discretization of the Laplace operator, and D is a diagonal matrix whose diagonal elements are just the values of p at the mesh points. It is not difficult to find that A is of the block tridiagonal form

$$A = \begin{bmatrix} G_1 & F_2 & & \\ E_2 & G_2 & \ddots & \\ & \ddots & \ddots & F_m \\ & & E_m & G_m \end{bmatrix},$$

with

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$$G_k = \begin{bmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}_{n_k \times n_k}, \quad k = 1, 2, \ldots, m,$$

$$E_k = F_k^T = \begin{cases} (O_{n_k \times p_k} \ -I_{n_k} \ O_{n_k \times q_k}), \\ \quad p_k, q_k \geqslant 0, \ p_k + q_k = n_{k-1} - n_k \\ \quad \text{if } n_k \leqslant n_{k-1}, \\ (O_{n_{k-1} \times s_k} \ -I_{n_{k-1}} \ O_{n_{k-1} \times t_k})^T, \\ \quad s_k, t_k \geqslant 0, \ s_k + t_k = n_k - n_{k-1} \\ \quad \text{if } n_k > n_{k-1}, \end{cases}$$

$$k = 2, 3, \ldots, m.$$

Obviously, as p is a sufficiently large positive number, the matrix $\mathcal{A}$ becomes highly indefinite and ill-conditioned.

As is known, the linear system (1.2) can be solved by direct methods or iterative methods. Direct methods are widely employed when the order of the coefficient matrix $\mathcal{A}$ is not too large, and are usually regarded as robust methods. The memory and the computational requirements for solving the large linear systems may seriously challenge the most efficient direct solution method available today.

The alternative is to use iterative methods established for solving the large sparse linear systems. Naturally, it is necessary that we make the use of iterative methods instead of direct methods to solve the large sparse linear systems. Meanwhile, iterative methods are easier to implement efficiently on high performance computers than direct methods. Currently, Krylov subspace methods are considered as one kind of the important and efficient iterative techniques for solving the large sparse linear systems. However, in fact, Krylov subspace methods are not competitive without a good preconditioner. In this Letter, positive stable preconditioners are presented to improve the convergence of Krylov subspace methods for solving Helmholtz equations.

There are the various authors contributed to the development of the powerful preconditioners for Helmholtz equations. In [3], Bayliss et al. equivalently transformed (1.2) into the normal equations $\mathcal{A}^H \mathcal{A}x = \mathcal{A}^H b$ and employed a preconditioned CG method to solve these normal equations. The preconditioning technique was based on an SSOR sweep for the discrete Laplace operator. Similar to the approach of [3], the preconditioner considered in [4] by Gozani et al. was constructed based on the Laplace operator, too. To make the above preconditioner better, in [5] Laird took the Laplace operator perturbed by a real-valued linear term as a preconditioner, which resulted in very satisfactory convergence. In [2], Erlangga et al. showed that making use of a complex-valued linear term to perturb the Laplace operator can lead to a better preconditioner than by using a real-valued perturbation. This kind of preconditioners is called "shifted Laplace" preconditioners, which is simple to construct and is easy to extend to inhomogeneous medias.

Based on the work of [2,5] and the preconditioning idea, this Letter is devoted to giving the positive stable preconditioners for the symmetric indefinite linear system (1.2).

The remainder of this Letter is organized as follows. In Section 2, iterative methods with the positive stable preconditioners for solving the resulting linear system will be discussed. In Section 3, the preconditioners will be extended to general mathematical model problems. In Section 4, numerical experiments are presented to confirm the efficiency of the presented preconditioners. Finally, in Section 5 some conclusions are drawn.

2. Preconditioned iterative methods

To improve the rate of convergence for iterative methods, in general, a preconditioner should be incorporated. That is, it is often preferable to solve the preconditioned linear system as follows:

$$P^{-1} \mathcal{A}x = P^{-1} b, \tag{2.1}$$

where P, called the preconditioner, is a non-singular matrix. The choice of the preconditioner P is important in actual implements. Generally speaking, the preconditioner P is chosen such that the condition number of the preconditioned matrix $P^{-1}\mathcal{A}$ is less than that of the original matrix $\mathcal{A}$. According to the survey of [6] by Benzi, a good preconditioner should meet the following requirements:

- The preconditioned system should be easy to solve.
- The preconditioner should be cheap to construct and apply.

Of course, the best choice for P^{-1} is the inverse of $\mathcal{A}$. However, it is useless in actual implements. If $\mathcal{A}$ is a symmetric positive definite matrix, the approximation of $\mathcal{A}^{-1}$ is taken place of SSOR or multi-grid. However, in fact, the Helmholtz equation results in an indefinite linear system, for which SSOR or multi-grid may be not guaranteed to converge.

To improve the convergence rate of iterative methods for solving the symmetric indefinite linear system arising from the Helmholtz equation, it is an easy approach that we may look for a form of P^{-1} such that $P^{-1}\mathcal{A}$ has satisfactory properties for Krylov subspace acceleration, not seek an approximate inverse of the indefinite matrix $\mathcal{A}$. In this way, a first effort to construct a preconditioner was presented in [3]. That is, the preconditioner is

$$P_1 = \Delta,$$

which is in connection with CGNR [7] to solve the symmetric indefinite linear system. One SSOR or multi-grid iteration is used whenever operations involving P_1^{-1} are required. The subsequent work on this class of the preconditioners with multi-grid was discussed in [4,13,14]. To make the above preconditioner better, in [5] Laird took the Laplace operator perturbed by a real-valued linear term as a preconditioner and improved the convergence rate of iterative methods. In [2], it was shown that making use of a complex-valued linear term to perturb the Laplace operator can lead to a better preconditioner than by using a real-valued perturbation.

As is known, Krylov subspace methods such as GMRES can be used to solve the preconditioned linear systems efficiently, too. In the case of small h, the storage problem of full GMRES can be overcome by applying GMRES(m). BiCGSTAB does not always perform satisfactorily in [5]. See [5] for more details.

To improve the preconditioner P_1 for the Helmholtz equation, here the new preconditioner is considered in terms of the idea of the preconditioner developed:

$$P_2 = A + h^2(\alpha - p)I \quad (\alpha > p). \tag{2.2}$$

Clearly, P_2 is a symmetric positive definite matrix. As before, the coefficient matrix in (1.2) is symmetric indefinite if p is a sufficiently large positive number. So, it is easy to know that MINRES or SYMMLQ with the preconditioner P_2 can be employed to solve the symmetric indefinite linear system (1.2), which was proposed in [8]. One can see [8] for details.

In our numerical experiments, we find that the MINRES method outperforms the SYMMLQ method, which are employed to solve the symmetric indefinite linear system (see our numerical experiments for details).