Physics Letters A 373 (2009) 2420–2427

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Entanglement of three-level trapped ions with phonon trap modes

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ARTICLE INFO

Article history:
Received 27 January 2009
Received in revised form 30 April 2009
Accepted 5 May 2009
Available online 9 May 2009
Communicated by P.R. Holland

PACS:
03.67.-a
42.50.-p

Keywords:
Three-level system
Entanglement

ABSTRACT

The entanglement between a single electron in the electronic states of a trapped three-level ion and the ionic vibrational modes of the trap is studied for an initially unentangled state of an electron level and a coherent phonon state. The effects of time-independent and time-dependent couplings are discussed.

1. Introduction

Recent advances in the dynamics of trapped ions indicate that a macroscopic observer can effectively control the dynamics as well as perform a complete measurement of the states of microscopic quantum systems [1–7]. One aspect of advances in control and measurement is the ability to make tests of various phenomena arising in quantum mechanics but otherwise absent in classical mechanics [3,8,9]. Some examples of such non-classical effects, based on the ability of quantum states to entangle their degrees of freedom, are found in the Einstein, Podolsky, Rosen paradox [10], discussions related to Bell's Theorem [11], and quantum teleportation [12]. Equally important aspects of the non-classical effects in quantum mechanics occur in potential technological applications of entanglement [6–9,13]. These include the use of entangled states in the design of quantum information and quantum computing systems [7–9,13], quantum teleportation and encryption [8,9,12,14], various processes that can regulate the propagation of light through optical media or the positioning of atoms in space [9], and aspects of entanglement-induced transparency [15]. As a consequence of these possible applications, entanglement has been studied in a wide variety of systems in attempts to find those that can sustain entanglement and efficiently utilize its properties [16–23].

A particular recent interest is the dynamics of trapped ions and the entanglement of the electronic and vibrational modes of the ion within the trap [3,6,7,13]. The interaction of the electronic and vibrational modes of a cooled trapped ion is mediated by an electric dipole coupling of the electronic levels of the ion to a classical electromagnetic field through an intensity gradient. Such systems find their roots in earlier studies of the spatial manipulation of atoms with laser beams [24], the laser cooling of atoms [25], and the study of Bose–Einstein condensates [26]. A technological focus of current work on these systems centers on the suggestion for their effective application as a means of developing entangled states that are suitable in the design of quantum computers. Much of these considerations both theoretically and experimentally have concentrated on the interaction of two-level ions with vibrational trap modes [6–9]. From a theoretical standpoint studies of these systems have included treatments of the ion interacting with Fock, coherent, and pair cat phonon states [6–8,27]. This has paralleled experimental work which has been involved in the preparation and measurement of the motional state of a trapped ion initially laser cooled to its zero-point of motion [6,7,28] and treatments of the interaction of two-level systems with Fock states, coherent states, squeezed states, and considerations of various side-band phonon transitions [19]. Of particular interest to us are recent experiments on two-level ions coupled to their trap vibrational modes [29] as these suggest that experiments can also be made on trapped three-level ions which are the topic of theoretical discussions here.

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2009.05.007

Indeed, the two-level experiments display many of the feature discussed in this Letter for three-level systems. An important aspect of much of this work is on the effective development of entangled systems having potential applications.

In this communication we extend considerations of the entanglement of two-level trapped ions with their phonon modes to treat a simple three-level trapped ion initial interacting with a coherent phonon state of its motion. In these considerations we will treat a system which represents the next level of complexity from that of the two-level system for a trapped ion and will determine the effects of this complexity on the system properties. The focus is on determining the entanglement properties of the system. Of interest are such properties as the rate at which a state of near maximum entanglement is achieved starting from initial conditions of a pure unentangled state, whether or not the system approaches a final state of near maximum entanglement asymptotically in time, whether or not a maximum entanglement is reached in a uniform manner, and how do time-dependent changes in the coupling of the ion and phonon fields affect the entanglement observed in the system.

It is important to point out that further insights into the dynamics of the multi-level systems may be helpful in developing quantum information theory [30–32]. It was demonstrated that key distributions based on three-level quantum systems are more secure against eavesdropping than those based on two-level systems [33]. The security of these protocols is related to the violation of the Bell inequality and a much smaller of noise can be tolerated using a three-level system [32]. Also, the use of qutrits rather than qubits can theoretically increase the speed of calculations. It appears therefore very tempting to investigate and compare the dynamics of the three-level system with those of two-level ones.

The presentation is organized as follows. In Section 2, the model of the three-level ion interacting with the field is presented. In Section 3, a discussion of the method of solution of the system dynamics and of the definition and evaluation of the von Neumann entropy is presented. In Section 4, the numerical results of entanglement and the generation and decay and revivals of entanglement are presented for the three-level system. Finally, a summary of the results and conclusions are given in Section 5 along with some generalization to higher multi-level systems with degeneracies.

2. System and model

The system to be studied is shown in Fig. 1. It consists of a trapped ion with a single electron restricted to occupy three electronic levels of the ion. The electron interacts with the phonons of the harmonic motion of the ion within the trap through the mediation of a classical electromagnetic field applied to the system [3–7]. The potential energy of the ionic motion in the trap is taken to be asymmetric such that the lowest frequency modes are for motion along the x-direction, and the system is cold so that the vibrations along the x-direction dominate the motion of the ion. The Hamiltonian of the ionic external and internal degrees of freedom is then given by

$$\hat{H}_0 = \hbar\upsilon\hat{\psi}^\dagger\hat{\psi} + \hbar\sum_{i=a,b,c}\omega_i\hat{S}_{ii}, \tag{1}$$

where $\hat{S}_{ij} = |i\rangle\langle j|$ $(i,j=a,b,c)$ are defined over the three electronic energy levels and $\omega_b - \omega_a = \omega_a - \omega_c$ where $\omega_b > \omega_a > \omega_c$. Here $\hat{\psi}$ and $\hat{\psi}^\dagger$ are the annihilation and creation operators of the harmonic modes of the ionic motion in the trap and υ is the frequency for motion along the x-direction. The higher frequency quantized vibrations in the y- and z-directions are ignored.

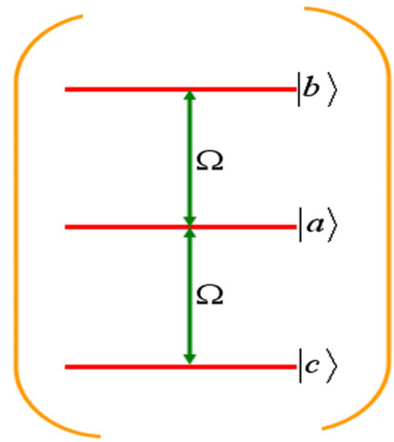

Fig. 1. Configuration of a single three-level ion. The states $|a\rangle$, $|b\rangle$ and $|c\rangle$ are coupled via a classical laser field of frequency Ω and a phonon of frequency υ so that the resonant transitions between the level are $\Omega + \upsilon$.

The interaction between the three-level ion and its harmonic motion is mediated by the electric dipole coupling of a classical electromagnetic field to the electronic states of the ion. The interaction of the three-level ion and the electric field is written as

$$\hat{H}_{\text{int}}(t) = -\Im(\hat{x},t)\cdot\hat{d}, \tag{2}$$

where $\hat{d} \propto \hat{S}_{ab} + \hat{S}_{ac} + \hat{S}_{ba} + \hat{S}_{ca}$ is the atomic electric dipole operator, and the classical electromagnetic field at the ion position is given by

$$\Im(\hat{x},t) = \Im\exp\left[-i(\hat{k}_{||}\cdot\hat{x} - \Omega t)\right] + \text{h.c.} \tag{3}$$

Here $\Im = \Im_0\exp(i\phi(t))$, where $\Im_0$ is the positive real constant electric field amplitude and $\phi(t)$ is a complex phase that can be used to model general time-dependent or stochastic couplings between the fields and dipole [34–37], $\hat{k}_{||}$ is the wave vector which is directed along the x-axis, $\hat{x}$ is the position of the ion center of mass operator, and Ω is the frequency of the laser field. Introducing the position operator $\hat{x} = [\hbar/2\upsilon M]^{1/2}(\psi + \psi^\dagger)$ into the factor of $\exp[i\hat{k}_{||}\cdot\hat{x}]$ in Eq. (3) gives $\exp[i\hat{k}_{||}\cdot\hat{x}] \propto \exp[i\eta(\psi^\dagger + \psi)]$ where $\eta = k_{||}(\hbar/2\upsilon M)^{1/2}$ with $0 < \eta \ll 1$ is the Lamb–Dicke parameter and M is the atomic mass [3]. (Here we assume that $0 < \eta \ll 1$ which is not unreasonable in realized systems [19,29].) Upon applying the Baker–Hausdorff theorem [38]

$$\exp\{i\eta(\psi^\dagger + \psi)\} = \exp\left(\frac{-\eta^2}{2}\right)\sum_{n=0}^{\infty}\frac{(i\eta)^n\psi^{\dagger n}}{n!}\sum_{m=0}^{\infty}\frac{(i\eta)^m\psi^m}{m!}, \tag{4}$$

and we see that the coupling to the electronic levels in powers of the Lamb–Dicke parameter can be divided into three categories: (i) the terms for $n > m$ correspond to an increase in energy linked with the motional state of center of mass of the ion by $(n-m)$ quanta, (ii) the terms with $n < m$ represent destruction of $(m-n)$ quanta of energy linked with the center of mass motion, and (iii) $(n = m)$ represents the diagonal contributions.

Using Eqs. (3) and (4) in Eq. (2), taking the Lamb–Dicke limit (i.e., $0 < \eta \ll 1$), and applying the rotating wave approximations [19], gives an interaction Hamiltonian of the form $(\hbar = 1)$

$$\hat{H}_{\text{int}} = \lambda_{ab}(t)\psi^\dagger\mathcal{E}_1^*(\psi^\dagger\psi)\hat{S}_{ab} + \lambda_{ca}(t)\psi^\dagger\mathcal{E}_1^*(\psi^\dagger\psi)\hat{S}_{ca} + \text{H.c.}, \tag{5}$$

where $\mathcal{E}_1(\psi^\dagger\psi)$ is given by

$$\mathcal{E}_1(\psi^\dagger\psi) = \exp\left(-\frac{\eta^2}{2}\right)\sum_{n=0}^{\infty}\frac{(i\eta)^{2n+1}}{n!(n+1)!}\psi^{\dagger n}\psi^n. \tag{6}$$