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Deltons, peakons and other traveling-wave solutions of a Camassa–Holm hierarchy

Xiaochun Peng [a,b], Hui-Hui Dai [b,*]

[a] *College of Mathematics and Statistics, Wuhan University, Wuhan 430072, PR China*
[b] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*

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ABSTRACT

In this letter, we study an integrable Camassa–Holm hierarchy whose high-frequency limit is the Camassa–Holm equation. Phase plane analysis is employed to investigate bounded traveling wave solutions. An important feature is that there exists a singular line on the phase plane. By considering the properties of the equilibrium points and the relative position of the singular line, we find that there are in total three types of phase planes. Those paths in phase planes which represented bounded solutions are discussed one-by-one. Besides solitary, peaked and periodic waves, the equations are shown to admit a new type of traveling waves, which concentrate all their energy in one point, and we name them deltons as they can be expressed as some constant multiplied by a delta function. There also exists a type of traveling waves we name periodic deltons, which concentrate their energy in periodic points. The explicit expressions for them and all the other traveling waves are given.

1. Introduction

The Camassa–Holm equation

$$\rho_t - \rho_{xxt} = \varepsilon\rho_x - 3\rho\rho_x + \rho\rho_{xxx} + 2\rho_x\rho_{xx}, \quad \varepsilon \in \mathbb{R},\ t > 0, \tag{1.1}$$

arises as a model for the unidirectional propagation of shallow water waves over a flat bottom, $\rho(x,t)$ representing the fluid velocity in the x-direction, and $\varepsilon \in \mathbb{R}$ being a constant related to the critical shallow water wave speed [1], which is proportional to the square root of the water depth. For $\varepsilon = 0$ and the constant wave speed $V \neq 0$, Camassa and Holm [1] showed that Eq. (1.1) has the peakon solution of the form $\rho(x,t) = Ve^{-|x-Vt|}$, which was given special attention in [2] because of its mathematical interest. The traveling waves of Camassa–Holm equation were further investigated recently in [3,4] and the references therein.

Johnson [5] questioned the validity of the first derivation of (1.1) given in the paper by Camassa and Holm [1] and provided a consistent derivation for (1.1) as a model equation in a shallow water. In the context of another valid physical model, Dai [6] (see also Dai and Huo [7]) derived the following model equation for nonlinear dispersive waves in cylindrical hyperelastic rods: $v_t + 3vv_x - v_{xxt} = \gamma(2v_xv_{xx} + vv_{xxx})$, where γ is a material parameter. Obviously, for $\gamma = 1$, this equation reduces to the Camassa–Holm equation with $\varepsilon = 0$. Constantin and Strauss [8] proved that the solitary waves of this model are orbitally stable for $\gamma \leqslant 1$; and for the case of $\gamma = 1$ they (Constantin and Strauss [9]) also gave the proof of the orbital stability of the peakons in the H^1 norm.

Camassa–Holm [1] also showed this equation was integrable and the isospectral problem was constructed. Eq. (1.1) can be determined from the spectral problem

$$\begin{cases} \psi_{xx} = (\frac{1}{4} + \frac{u(x,t)}{\lambda})\psi, \\ \psi_t = (\lambda - \rho(x,t))\psi_x + \frac{1}{2}\rho_x\psi, \end{cases} \tag{1.2}$$

* Corresponding author. Tel.: +852 2788 8660; fax: +852 2788 8561/2788.
E-mail address: mahhdai@math.cityu.edu.hk (H.-H. Dai).

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where $u = \frac{1}{2}(\rho - \rho_{xx}) - \frac{1}{4}\varepsilon$. In this letter, we consider the complete Camassa–Holm hierarchy. From the over-determined spectral problem

$$\begin{cases} \psi_{xx} = (\frac{1}{4} + \frac{u(x,t)}{\lambda})\psi, \\ (\lambda - \zeta)\psi_t = a(x,t,\zeta)\psi_x - \frac{1}{2}a_x(x,t,\zeta)\psi, \end{cases} \tag{1.3}$$

which is first introduced in [10], we immediately can obtain the Camassa–Holm hierarchy

$$\begin{cases} u_t = \frac{1}{2}(a_x - a_{xxx}), \\ \frac{1}{2}\zeta(a_{xxx} - a_x) = 2ua_x + au_x. \end{cases} \tag{1.4}$$

Dai and Pavlov [10] showed that its high-frequency limit is the Camassa–Holm equation and low-frequency limit is the Hunter–Saxton equation, which is a model for the motion of a nematic liquid crystal (see [11]).

We also point out that traveling waves, whether their solution expressions are in explicit or implicit forms, are very interesting from the point of view of applications. These types of waves will not change their shapes during propagation and are thus easy to detect. Recently, besides the previously known smooth, peaked, and cusped waves, some exotic waves such as compactons, stumpons, and fractal-like waves are given special attention. In [12,13], it was shown that some dispersive nonlinear wave equations admit some exotic solutions. In this letter, phase plane analysis is employed to investigate the bounded traveling wave solutions of Eqs. (1.4) for $a(x,t)$, then $u(x,t)$ is easily obtained from $(1.4)_1$, and two new types of exotic waves for $u(x,t)$ are found.

An important feature of this system is that there exists a singular line on the phase plane evolving the complexity of the solution. It was first found by Dai [14] that the importance of such a singular line is it can cause the appearance of a variety of singular wave patterns including peakons [6,7]. By considering the properties of the equilibrium points and the relative position of the singular line, we find that there are in total three types of phase planes. Those paths in phase planes which represent bounded solutions are discussed one-by-one. It turns out that Eqs. (1.4) for $a(x,t)$ admit a variety of traveling waves, including solitary waves, peakons, periodic peakons and periodic waves, and the analytical expressions for those traveling waves are obtained. And then the expressions for $u(x,t)$ are also obtained.

We find that the traveling waves for $u(x,t)$ are the same type as that of $a(x,t)$ when $a(x,t)$ be solitary waves and periodic waves. However, very interestingly, we demonstrate that when $a(x,t)$ be peakons Eqs. (1.4) generate a type of new solitary waves for $u(x,t)$. Not like the compactons [15] whose energy concentrates in a finite interval, this new type of traveling waves concentrate all their energy at one point. And we name them deltons as they can be expressed as some constant multiplied by a delta function. It is also found that the solution for $u(x,t)$ are periodic deltons when $a(x,t)$ be periodic peakons.

2. Phase-plane analysis

Now we consider the traveling wave solutions of Eqs. (1.4). Substituting $u(x,t) = u(\xi)$, $a(x,t) = a(\xi)$, and $\xi = x - Vt$ (V is the propagation speed) into (1.4), we have

$$\begin{cases} -Vu_\xi = \frac{1}{2}(a_\xi - a_{\xi\xi\xi}), \\ \frac{1}{2}\zeta(a_{\xi\xi\xi} - a_\xi) = 2ua_\xi + au_\xi. \end{cases} \tag{2.1}$$

It is easily verified that the bounded solution is trivial when $V = 0$. In fact, integrating (2.1) directly, we obtain $u(\xi) = c_1(c_2e^{\xi} + c_3e^{-\xi} + c_4)^2$, where c_1, c_2, c_3, c_4 are integration constants. Thus, in this case only the constant solution is bounded.

Therefore, we only focus on the case of $V \neq 0$. Integrating $(2.1)_1$, we have

$$-Vu + d_1 = \frac{1}{2}(a - a_{\xi\xi}), \tag{2.2}$$

where d_1 is an integration constant. Substituting (2.2) into $(2.1)_2$, we have

$$-(V\zeta + 4d_1)a_\xi + V\zeta a_{\xi\xi\xi} = -3aa_\xi + 2a_\xi a_{\xi\xi} + aa_{\xi\xi\xi}. \tag{2.3}$$

Integrating (2.3) twice, we obtain

$$a_\xi^2 = F(a, d_3), \tag{2.4}$$

where $F(a,d_3) = \frac{-2d_3 - 2d_2a + (V\zeta + 4d_1)a^2 - a^3}{V\zeta - a}$, d_2 and d_3 are two integration constants. Mathematically, (2.4) will yield a solution when a group of values of $(V, \zeta, d_1, d_2, d_3)$ is given. However, that solution may not be bounded. As pointed out before, we are only interested in the bounded solutions for $a(x,t)$. Next, we shall examine for which values of $(V, \zeta, d_1, d_2, d_3)$ Eq. (2.4) yields bounded solutions and how many types of traveling waves exist. For this purpose, we shall conduct a phase-plane analysis.

Let y denote a_ξ, then from (2.4) d_3 can be denoted by $d_3 = g(a, y)$. Differentiating (2.4) once with respect to ξ gives the following first-order system:

$$\begin{cases} a_\xi = y, \\ y_\xi = \frac{1}{2}F'(a, g(a,y)), \end{cases} \tag{2.5}$$

where the prime denotes differentiation with respect to a. It is easily verified that the first coordinate of the equilibrium points of system (2.5) is governed by

$$3a^2 - 2(V\zeta + 4d_1)a + 2d_2 = 0. \tag{2.6}$$

We can see that a phase plane is four-parameter (V, ζ, d_1 and d_2) dependent, and when d_3 takes different values, (2.4) yields different orbits in the phase plane.