Physics Letters A 373 (2009) 2484–2487

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Singularities motion equations in 2-dimensional ideal hydrodynamics of incompressible fluid

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ARTICLE INFO

Article history:
Received 10 December 2008
Received in revised form 6 February 2009
Accepted 10 February 2009
Available online 14 February 2009
Communicated by V.M. Agranovich

PACS:
47.10.+g
47.32.C-
47.32.cb

Keywords:
Ideal hydrodynamic
Point vortice

ABSTRACT

In this Letter, we have obtained motion equations for a wide class of one-dimensional singularities in 2D ideal hydrodynamics. The simplest of them, are well known as point vortices. More complicated singularities correspond to vorticity point dipoles. It has been proved that point multipoles of a higher order (quadrupoles and more) are not the exact solutions of two-dimensional ideal hydrodynamics. The motion equations for a system of interacting point vortices and point dipoles have been obtained. It is shown that these equations are Hamiltonian ones and have three motion integrals in involution. It means the complete integrability of two-particle system, which has a point vortex and a point dipole.

1. In 2-dimensional ideal hydrodynamics of incompressible fluid, point vortices play an exceptionally important role. Actually, point vortices are the well-defined quasiparticles. This means that various flows of ideal fluid can be represented as movements induced by a system of interacting point vortices [1,2]. Certain restrictions of this point of view are discussed, in particular, in [3]. It is well known that the dynamics of three point vortices is regular while four or more point vortices dynamics is chaotic [4–7]. The generalization of such point vortices on sphere or in isolated areas was already done by some authors [8,9] (see also [2,10]). Point vortices are also used for two-dimensional turbulence models [11–13] and for studies of 2D turbulence spectra. The nature of vortices interaction and features of statistically stationery states generation are essential [14,15], as well as the nature of singular velocity field, induced by point vortices movements [16]. Besides, point vortices engender more complicated solutions of 2D Euler equation and magnetic fields configuration in MHD [17–19]. In this Letter, we will show that 2D Euler equation has exact solutions with more complicate singularities, such as a set of point vortex dipoles. Any finite sum of sets of point vortex dipoles and of point vortices is also an exact solution for 2D Euler equation with the moving singularities. Motion equations for a system of interacting vortices and point dipoles have been obtained. It is shown that these equations are Hamiltonian ones and have three motion integrals in involution.

2. In this part, we are going to obtain motion equations of different kind of one-dimensional singularity for two-dimensional Euler equations. Let us start with two-dimensional case of Euler equation for incompressible fluid:

$$\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\frac{\partial P}{\partial x_i}, \quad \operatorname{div} \vec{V} = 0. \tag{1}$$

Now we shall use the potential φ of velocity field $\vec{V}$, which is defined as:

$$V_i = \varepsilon_{ik} \frac{\partial \varphi}{\partial x_k}. \tag{2}$$

Here $i = 1, 2$ and ε_{ik} is a unit antisymmetric tensor. After excluding pressure we shall consider a well-known form of Euler equation:

$$\frac{\partial \Delta\varphi}{\partial t} + \{\Delta\varphi, \varphi\} = 0. \tag{3}$$

Here Δ is two-dimensional Laplace operator, and $\{A, B\} = \varepsilon \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_k}$ is Poisson's bracket. From physical point of view, this means the freezing of vorticity field ω into the fluid. Now the vorticity ω can be defined as:

$$\omega = -\Delta\varphi. \tag{4}$$

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2009.02.023

Let us suppose, that vorticity field singularities can be defined in terms of generalized functions as:

$$-\Delta\varphi = \sum_{\alpha=1}^{N} \Gamma_\alpha \delta\big(\vec{x} - \vec{x}_v^\alpha(t)\big) + \sum_{\beta=1}^{M} D_m^\beta(t) \frac{\partial \delta(\vec{x} - \vec{x}_d^\beta(t))}{\partial x_m} + \sum_{\gamma=1}^{K} \mu_{i_1 i_2}^\gamma(t) \frac{\partial^2 \delta(\vec{x} - \vec{x}_\mu^\gamma(t))}{\partial x_{i_1} \partial x_{i_2}} + \cdots. \tag{5}$$

In other words, vorticity field is represented as a sum of point multipoles and point vortices. Greek indexes in the right part numerate the corresponding objects. The first sum corresponds to point vortices; Γ_α is vortex stretch and $\vec{x}_v^\alpha(t)$ is the coordinate of αth vortex. The second sum corresponds to point dipoles; $D_m^\beta(t)$ is dipole moment and $\vec{x}_d^\beta(t)$ is the coordinate of βth point dipole. The next contributions correspond to multipoles of a higher order, e.g. $\mu_{i_1 i_2}^\gamma(t)$ is quadrupole moment (symmetric traceless tensor) and $\vec{x}_\mu^\gamma(t)$ is the coordinate of γth point quadrupole. The sense of the designation is the same as described above.

It is easy to obtain the explicit form of the potential from Eq. (5)

$$\varphi = -\frac{1}{4\pi} \sum_{\alpha=1}^{N} \Gamma_\alpha \ln\big|\vec{x} - \vec{x}_v^\alpha(t)\big| - \frac{1}{2\pi} \sum_{\beta=1}^{M} D_l^\beta(t) \frac{(x_l - x_{ld}^\beta(t))}{|\vec{x} - \vec{x}_d^\beta(t)|^2} - \frac{1}{4\pi} \sum_{\gamma=1}^{N} \mu_{i_1,i_2}^\gamma(t) \frac{\partial^2 \ln|\vec{x} - \vec{x}_\mu^\gamma(t)|}{\partial x_{i_1} \partial x_{i_2}} \cdots. \tag{6}$$

The main problem is to find under which conditions Eq. (6) will be compatible with Euler equation (3). In order to solve this problem, we have to substitute the development (6) directly into Euler equation (3). Such substitution of Eqs. (5) and (6) into Eq. (3) leads to the following:

$$\sum_{\alpha=1}^{N} \Gamma_\alpha \left(\frac{dx_{vi}^\alpha(t)}{dt} - V_i \right) \frac{\partial \delta(\vec{x} - \vec{x}_v^\alpha(t))}{\partial x_i} - \sum_{\beta=1}^{M} \frac{dD_i^\beta(t)}{dt} \cdot \frac{\partial \delta(\vec{x} - \vec{x}_d^\beta(t))}{\partial x_i} - \sum_{\beta=1}^{M} D_m^\beta(t) \left(\frac{dx_{di}^\beta(t)}{dt} - V_i \right) \frac{\partial^2 \delta(\vec{x} - \vec{x}_d^\beta(t))}{\partial x_m \partial x_i} - \sum_{\gamma=1}^{N} \frac{d\mu_{i_1,i_2}^\gamma(t)}{dt} \cdot \frac{\partial^2 \delta(\vec{x} - \vec{x}_\mu^\gamma(t))}{\partial x_{i_1} \partial x_{i_2}} - \sum_{\gamma=1}^{N} \mu_{i_1,i_2}^\gamma(t) \left(\frac{dx_{\mu i}^\gamma(t)}{dt} - V_i \right) \frac{\partial^3 \delta(\vec{x} - \vec{x}_\mu^\gamma(t))}{\partial x_{i_1} \partial x_{i_2} \partial x_i} + \cdots = 0. \tag{7}$$

Here V_i means the components of velocity field calculated with Eqs. (6) and (2). To make it less bulky, we shall write down the dipole and quadrupole contributions only. There is no special difficulties in taking into account the contributions of higher order multipoles. Let us note that δ-functions in Eq. (7) are the functions of both t, and $\vec{x}$. Now we shall deal with Eq. (7) using properties of the generalized functions. In particular, the following well-known properties of δ-functions derivatives will be very important (see, for example, [20,21]):

$$\alpha(\vec{x}) \frac{\partial \delta(\vec{x} - \vec{A})}{\partial x_i} = \alpha(\vec{x})\big|_{\vec{x}=\vec{A}} \frac{\partial \delta(\vec{x} - \vec{A})}{\partial x_i} - \frac{\partial \alpha(\vec{x})}{\partial x_i}\bigg|_{\vec{x}=\vec{A}} \delta(\vec{x} - \vec{A}) \tag{8}$$

and

$$\alpha(\vec{x}) \frac{\partial^2 \delta(\vec{x} - \vec{A})}{\partial x_i \partial x_j} = \alpha(\vec{x})\big|_{\vec{x}=\vec{A}} \frac{\partial^2 \delta(\vec{x} - \vec{A})}{\partial x_i \partial x_j} - \frac{\partial \alpha(\vec{x})}{\partial x_i}\bigg|_{\vec{x}=\vec{A}} \frac{\partial \delta(\vec{x} - \vec{A})}{\partial x_j} - \frac{\partial \alpha(\vec{x})}{\partial x_j}\bigg|_{\vec{x}=\vec{A}} \frac{\partial \delta(\vec{x} - \vec{A})}{\partial x_i} + \frac{\partial^2 \alpha(\vec{x})}{\partial x_i \partial x_j}\bigg|_{\vec{x}=\vec{A}} \delta(\vec{x} - \vec{A}). \tag{9}$$

When using these relations, we can obtain δ-functions derivatives coefficients, depending only on time. Then, Eq. (7) takes the form:

$$\sum_{\alpha=1}^{N} \Gamma_\alpha \left(\frac{dx_{vi}^\alpha(t)}{dt} - V_i|_{\vec{x}=\vec{x}_v^\alpha(t)} \right) \frac{\partial \delta(\vec{x} - \vec{x}_v^\alpha(t))}{\partial x_i} - 2\sum_{\beta=1}^{M} \left(\frac{dD_i^\beta(t)}{dt} - D_m^\beta(t) \frac{\partial V_i}{\partial x_m}\bigg|_{\vec{x}=\vec{x}_d^\beta(t)} \right) \frac{\partial \delta(\vec{x} - \vec{x}_d^\beta(t))}{\partial x_i} + 2\sum_{\beta=1}^{M} D_m^\beta(t) \left(\frac{dx_{di}^\beta(t)}{dt} - V_i|_{\vec{x}=\vec{x}_d^\beta(t)} \right) \frac{\partial^2 \delta(\vec{x} - \vec{x}_d^\beta(t))}{\partial x_m \partial x_i} + \sum_{\gamma=1}^{N} \left(\frac{d\mu_{i_1,i_2}^\gamma(t)}{dt} - \mu_{i_3,i_2}^\gamma(t) \left(\frac{\partial V_{i_1}(\vec{x})}{\partial x_{i_3}}\bigg|_{\vec{x}=\vec{x}_\mu^\gamma(t)} \right) - \mu_{i_1,i_3}^\gamma(t) \left(\frac{\partial V_{i_2}(\vec{x})}{\partial x_{i_3}}\bigg|_{\vec{x}=\vec{x}_\mu^\gamma(t)} \right) \right) \cdot \frac{\partial^2 \delta(\vec{x} - \vec{x}_\mu^\gamma(t))}{\partial x_{i_1} \partial x_{i_2}} - \sum_{\gamma=1}^{N} \mu_{i_1,i_2}^\gamma(t) \left(\frac{dx_{\mu i_3}^\gamma(t)}{dt} - \big(V_{i_3}(\vec{x})|_{\vec{x}=\vec{x}_\mu^\gamma(t)}\big) \right) \cdot \frac{\partial^3 \delta(\vec{x} - \vec{x}_\mu^\gamma(t))}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} + \sum_{\gamma=1}^{N} \mu_{i_1,i_2}^\gamma(t) \left(\frac{\partial^2 V_{i_3}(\vec{x})}{\partial x_{i_1} \partial x_{i_2}}\bigg|_{\vec{x}=\vec{x}_\mu^\gamma(t)} \right) \cdot \frac{\partial \delta(\vec{x} - \vec{x}_\mu^\gamma(t))}{\partial x_{i_3}} + \cdots = 0. \tag{10}$$

From (10) we can obtain the motion equation for interacting singularities and the evolution equations for multipole moments. In order to do it, we need to have all the coefficients equal to zero before different δ-functions derivatives independently. If all these equations are compatible, then the velocity field, generated by potential (6), is the exact generalized solution of two-dimensional Euler equation. It is easy to see that if coefficients before generalized functions relative to vortexes and dipoles tend to zero, then it gives vortex and dipoles motion equations only, as well as evolution law for dipole moment. Further, it will be shown that equations of this set are compatible. But the situation is getting substantially different starting from the quadrupole vortices. As a matter of fact, higher order multipole moments give a set of uncompatible equations. In general case, the compatibility conditions are satisfied only, if higher order multipole moments starting from quadrupole one, are zero. Hence, in two-dimensional hydrodynamics only sets of point vortices and point dipoles give generalized point solutions. Let us note that singularities motion equations have natural physical meaning of the vorticity freezing into medium motions. Thus, the motion velocity of a chosen singularity coincides with the medium velocity in the same point, induced by all other singularities. From mathematical point of view, the selfinteraction is absent in motion equations for dipole singularities as well as for point vortices. Unlike point vortices when vortex stretch is constant, the dipole moment is function of time. Let us give now the final equations for evolution of interacting point vortices and point dipoles:

$$\frac{d\vec{x}_{vi}^\alpha(t)}{dt} = -\varepsilon_{ik} \left\{ \sum_{\gamma \neq \alpha}^{N} \frac{\Gamma_\gamma}{2\pi} \frac{(x_{vk}^\alpha - x_{vk}^\gamma)}{|\vec{x}_v^\alpha - \vec{x}_v^\gamma|^2} + \sum_{\beta}^{M} \frac{D_l^\beta(t)}{\pi} \left(\frac{\delta_{lk}}{|\vec{x}_v^\alpha - \vec{x}_d^\beta|^2} - \frac{2(x_{vl}^\alpha - x_{dl}^\beta)(x_{vk}^\alpha - x_{dk}^\beta)}{|\vec{x}_v^\alpha - \vec{x}_d^\beta|^4} \right) \right\} = 0, \tag{11}$$