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Confidence bounds of recurrence-based complexity measures

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ABSTRACT

In the recent past, recurrence quantification analysis (RQA) has gained an increasing interest in various research areas. The complexity measures the RQA provides have been useful in describing and analysing a broad range of data. It is known to be rather robust to noise and nonstationarities. Yet, one key question in empirical research concerns the confidence bounds of measured data. In the present Letter we suggest a method for estimating the confidence bounds of recurrence-based complexity measures. We study the applicability of the suggested method with model and real-life data.

1. Introduction

Recurrence Plots (RP) and their quantification (*recurrence quantification analysis*, RQA) [11] have become rather popular in various fields of science. The complexity measures based on RPs have helped to gain a deeper insight into diverse kinds of phenomena and experimental data. In this Letter we propose a straightforward extension to the existing RQA framework which allows us to not only compute these complexity measures, but also to estimate their confidence bounds. We do this by using a well-known resampling paradigm – the bootstrap. We show that the confidence bounds of RQA measures come with the regular analysis at virtually no extra costs and that the method can be useful for comparing univariate time series in a statistically sound fashion.

2. Recurrence Plots and their quantification

Recurrence is a fundamental property of dynamical systems. On this basis the data analysis tool called Recurrence Plot (RP) has been devised by Eckmann et al. [1] which visualises recurrences in the phase space of an n-dimensional state vector $\vec{x}_i$ $(i = 1, \ldots, N)$,

$$R_{i,j} = \Theta\left(\varepsilon - \|\vec{x}_i - \vec{x}_j\|\right), \tag{1}$$

where Θ is the Heaviside function, $\|\cdot\|$ is a norm and ε is the recurrence threshold. The threshold ε can be defined as an absolute value or in dependence on other criteria. For the examples in Fig. 1 we chose ε so that the overall RR, Eq. (2), is 10%. The binary $N \times N$ matrix allows for a 2-dimensional visualisation of an n-dimensional attractor's recurrence properties (Fig. 1). From this matrix a number of well-defined complexity measures can be extracted (see [11] and references therein). If only univariate time-series are available the state vectors can be reconstructed using delay embedding with a given embedding dimension m and a delay τ [14,19].

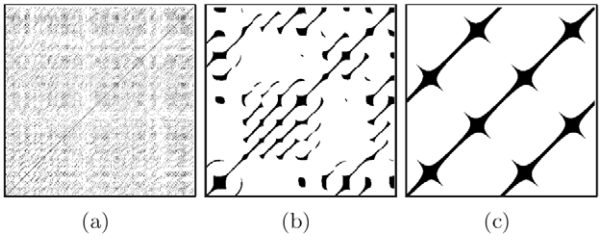

Fig. 1. Sample RPs for different dynamical systems. (a) Gaussian white noise, (b) the Lorenz attractor with $\rho = 28$, $\beta = 8/3$, $\sigma = 10$ and (c) a sine wave. Parameters were $m = 3$, $\tau = 3$, ε adjusted to $RR = 0.1$.

2.1. RQA measures

The information contained in an RP can be quantified by measures of complexity based on recurrence point density, *diagonal*

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and *vertical* line structures. The simplest measure is the *recurrence rate RR*,

$$RR = \frac{1}{N^2}\sum_{i,j} R_{i,j}, \tag{2}$$

which denotes the overall probability that a certain state recurs. A measure based on the distribution of *diagonal* structures $P(l)$ is the *determinism DET*

$$DET = \frac{\sum_{l\geqslant l_{\min}}^{l_{\max}} lP(l)}{\sum_{l=1}^{l_{\max}} lP(l)}, \tag{3}$$

the ratio of recurrence points in diagonal lines (of at least length $l_{\min}$) to all recurrence points. *DET* reflects how predictable a system is. The measures $L_{\max}$ and $\langle L\rangle$ are the maximum and average lengths of the diagonal lines in $P(l)$.

Further complexity measures quantify the *vertical* structures in an RP. The ratio of points forming vertical line structures of a minimal length $v_{\min}$ to all recurrence points is called *laminarity*

$$LAM = \frac{\sum_{v\geqslant v_{\min}}^{v_{\max}} vP(v)}{\sum_{v=1}^{v_{\max}} vP(v)}, \tag{4}$$

a measure sensitive to laminar states and regimes of intermittency. From the distribution of the vertical line structures $P(v)$ we can again compute the maximal vertical line length $V_{\max}$ and the average called *trapping time TT*.

These measures can be computed from the whole RP or in moving, possibly overlapping, windows of size w shifted along the main diagonal of the RP by an increment of s. This approach is useful to reveal qualitative transitions in a system.

The RQA measures provide a qualitative description of a system in terms of complexity measures. It allows to detect transitions in the system's dynamics, e.g. transitions from period to chaos, from strange nonchaotic attractors (SNA) to chaos or even transitions from chaos to chaos [9,13].

In the next section we will focus on how to derive a quantitative judgement from these measures.

3. Confidence bounds of RQA measures

The RQA measures have been quite useful for the analysis of a variety of data. Yet, in order to not only detect qualitative changes in a system's dynamics but to be able to judge their significance or to compare two univariate time series, it is necessary to derive a quantitative judgement such as a confidence interval. For recurrence-based complexity measures those intervals can be estimated using a resampling paradigm.

3.1. Resampling statistics – the bootstrap

Statistical techniques based on resampling were among the first methods ever thought of. Sir R.A. Fisher himself introduced this idea when pondering over Gosset's t-distribution [3,4]. Due to lacking computational power, these ideas were not feasible at that time. With the advent of powerful, low-cost computers these methods have gained a broad interest and have been proven to be very reliable and powerful. In this Letter we focus on one particular resampling method – the bootstrap [2]. The bootstrap is a nonparametric method for estimating the variance of a statistic of interest. It relies on resampling of a given distribution *with* replacement and does not require any specific probability distribution. The bootstrap procedure works as follows:

Given a random sample x_i $(i = 1, 2, \ldots, n)$ of size n, from an unspecified probability distribution we compute a statistic of interest, say, the mean $\langle x\rangle$. In order to estimate the variance of that statistic we draw at random and *with* replacement the same number (n) of elements from x_i to obtain the resampled distribution x_i^*. From x_i^* we again compute the statistic of interest. *With* replacement means that we can draw the individual elements in x_i more than once. Doing this a larger number of times[1] we obtain the empirical distribution of the statistic of interest, $\hat{P}_{\langle x\rangle}$. From the empirical distribution we can compute the percentiles $\alpha/2$ and $1-\alpha/2$ and define the $(100-\alpha)\%$ confidence interval (CI) as the range between those two percentiles.

The empirical distribution could also be used to perform hypothesis testing. We opt for the estimation of confidence intervals only. The interpretation of hypothesis tests, especially p-values, the chosen indicator of significance, is currently under discussion and not agreed upon by the frequentist and Bayesian schools. Therefore we follow the suggestions of Hubbard and Lindsay [6] and only estimate the confidence intervals of the RQA measures. This allows us to not only detect transitions in the dynamics of one system or to differences between the dynamics of two systems but to provide a judgement whether those differences are statistically significant. While this is not statistical testing in the narrowest sense, we think this approach is more appropriate as the data is explicitly shown and the investigator may judge for him-/herself.

3.2. Structure preserving resampling

Since the bootstrap relies on resampling *with* replacement we cannot simply bootstrap the RP matrix as such for two reasons. First of all, we could draw one of the black points more than once. As the RP is a binary matrix by definition this is not possible. Secondly, randomly resampling an RP would necessarily result in a loss of most of the small-scale structures in it (i.e. diagonal and vertical lines). A loss of structures would result in an RP corresponding to noise. This is not desirable because we want to compare different systems against each other and not test against randomness/noise.

As stated above, RQA measures like *DET* or *LAM* rely on the distribution of line structures $P(l)$ and $P(v)$. Therefore we present a method that ensures that the structural elements are preserved during resampling. We only resample the distributions of diagonal and vertical lines, $P(l)$ and $P(v)$. It is important to note that we need to resample all lines in $P(l)$ and $P(v)$, even those of only length 1, thereby obtaining $P^*(l)$ and $P^*(l)$, respectively. The value of determinism is then computed according to:

$$DET^{*i} = \frac{\sum_{l\geqslant l_{\min}}^{l_{\max}} lP^{*i}(l)}{\sum_{l=1}^{l_{\max}} lP^{*i}(l)} \quad (i = 1, 2, \ldots, n_{bs}) \tag{5}$$

for each bootstrapped sample (see Fig. 2). The computation for a bootstrapped sample of $\langle L\rangle^*$, LAM^* and TT^* is done accordingly. Repeating this procedure n_{bs} times we obtain the empirical distributions $\hat{P}_{DET}$, $\hat{P}_{\langle L\rangle}$, $\hat{P}_{LAM}$ and $\hat{P}_{TT}$. From the empirical distributions we can calculate the percentiles $\alpha/2$ and $1-\alpha/2$. The two-sided $(100-\alpha)\%$ confidence interval is then defined as the range between those two percentiles. The value α determines the spread of the interval, the smaller α, the broader the interval. As we leave the structures in the RP intact, we refer to this procedure as *structure preserving resampling*.

For obvious reasons this approach is restricted to *DET*, $\langle L\rangle$, *LAM* and *TT*. $V_{\max}$ and $L_{\max}$ already represent maxima in the distribution and are therefore very unlikely to show variation and the upper bound cannot vary at all. Furthermore, we can apply this

[1] The number of resamplings is not generally agreed upon but common guidelines suggest values between 800 and 1500. Note also that the number of resampling decreases with the *randomness* of the samples drawn. In the present manuscript we use the MT19937 algorithm, which has a period of $2^{19937}-1$ [12].